Higher gradients and microinertia effects on the dynamics of heterogeneous media

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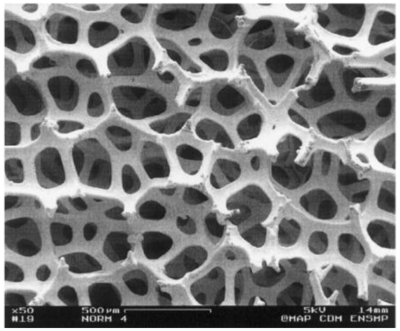
Motivations & scientific locks

• Dynamic models for composites prone to strain gradients and microinertia effects.

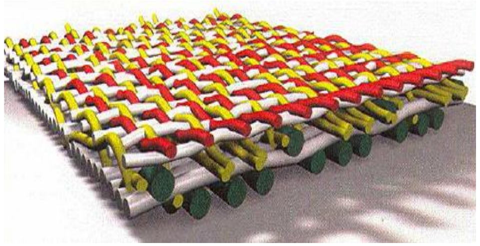
• Generalized continuum theories attractive alternative to capture dynamic behaviors overlooked by classical elasticity, like dispersive wave behavior due to microstructural effects (Lombardo and Askes, 2012; Ostoja-Starzewki, 2002; Papargyri-Beskou and Beskos, 2004; Askes and Aifantis, 2011).

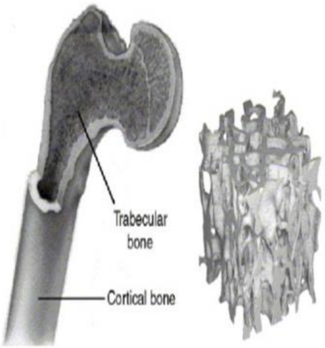
• Set up homogenization framework to derive the effective dynamical enriched behavior

• Applications to composites: case of laminates



polymeric foams





Foam-like structures

Dynamical Hill-Mandel principle for strain gradient mechanics with micro-inertia

$\sigma_{ij,j} = \rho \ddot{u}_i$ Microscopic equilibrium

Average kinematics & statics of the strain gradient effective continuum

$$E_{ij} = \frac{1}{|\Omega|}\int_{\Omega} \varepsilon_{ij} d\Omega = \langle \varepsilon_{ij} \rangle = E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$$ Strain

$$K_{ijk} = \frac{1}{|\Omega|}\int_{\Omega} (\varepsilon_{ij} \otimes \nabla_k) d\Omega = \langle \varepsilon_{ij} \otimes \nabla_k \rangle = \frac{1}{2}\left(\frac{\partial^2 U_i}{\partial X_j \partial X_k} + \frac{\partial^2 U_j}{\partial X_i \partial X_k}\right)$$ Strain gradient

$$\Sigma_{ij} = \frac{1}{|\Omega|}\int_{\Omega} \sigma_{ij} d\Omega = \langle \sigma_{ij} \rangle, \qquad S_{ijk} = \langle \sigma_{ij} \otimes x_k \rangle$$ Stress and hyperstress

$$\int_{\partial\Omega} \left(u_i - x_j E_{ij} - \frac{1}{2} K_{ijk} x_j x_k\right)(\sigma_{mi} - \Sigma_{mi}).n_m ds = 0 \longrightarrow$$

$$E_{ij}\Sigma_{ij} + K_{ijk}S_{ijk} = \langle \varepsilon_{ij}\sigma_{ij} \rangle + \left\langle \rho \ddot{u}_i \left(u_i - E_{ij}x_j - \frac{1}{2}K_{ijk}x_j x_k\right)\right\rangle$$

Dynamical Hill-Mandel type relation with microinertia

Hill-Mandel principle for strain gradient mechanics with micro-inertia

Split displacement into homogeneous part + fluctuation

$$u_i = U_i^0 + \hat{u}_i, \forall x \in \Omega$$

$$u_i = U_i^0 = E_{ij}x_j + \frac{1}{2}K_{ijk}x_j x_k, \forall x \in \partial\Omega$$

Homogeneous part

$$u_i(X,x) = H^E{}_{ijk}(x)E_{jk}(X) + H^K{}_{ijkl}(x)K_{jkl}(X)$$

$$\rightarrow \varepsilon_{ij}(X,x) = A^E{}_{ijkl}(x)E_{kl}(X) + A^K{}_{ijklm}(x)K_{klm}(X)$$

Microscopic displacement versus mesoscopic kinematic variables using localizators

-> Mesoscopic dynamical equ. of motion with **static stress and hyperstress tensors**:

$$\Sigma_{ij,j} - S_{ijk,kj} + F_i = \langle\rho\rangle\ddot{U}_i$$

$$\Sigma_{ij} = \frac{1}{2}\left\langle A^E_{krij}C_{krmn}A^E_{mndl} + A^E_{rsdl}.C_{rsmn}.A^E_{mnij}\right\rangle E_{dl} + \frac{1}{2}\left\langle A^E_{klij}C_{klmn}A^K_{mnrst} + A^K_{klrst}C_{klmn}A^E_{mnij}\right\rangle K_{rst}$$

$$S_{ijk} = \frac{1}{2}\left\langle A^E_{rsdl}C_{rsmn}A^K_{mnijk} + A^K_{rsijk}C_{rsmn}A^E_{mndl}\right\rangle E_{dl} + \frac{1}{2}\left\langle A^K_{pfijk}C_{pfmn}A^K_{mnrst} + A^K_{dlrst}C_{dlmn}A^K_{mnijk}\right\rangle K_{rst}$$

Average body forces includes dynamical terms accounting for the microstructure

$$F_i = \frac{\partial}{\partial X_j}\left(\frac{1}{2}\ddot{E}_{mn}\left\langle\rho\left(H^E_{imn}x_j + H^E_{mij}x_n - 2\delta_i^m x_j x_n\right)\right\rangle + \frac{1}{2}\ddot{K}_{rsk}\left\langle\rho\left(H^K_{irsk}x_j + \frac{1}{2}H^E_{rij}x_s x_k - \delta_i^r x_s x_j x_k\right)\right\rangle\right)$$

$$-\frac{\partial^2}{\partial X_j \partial X_k}\left(\ddot{E}_{rs}\left\langle\frac{\rho}{2}\left(H^K_{rijk}x_s + \frac{1}{2}H^E_{irs}x_j x_k - \delta_i^p x_s x_j x_k\right)\right\rangle + \frac{1}{4}\ddot{K}_{rst}\left\langle\rho\left(H^K_{rijk}x_s x_t + H^K_{irst}x_j x_k - \delta_i^r x_s x_j x_k x_t\right)\right\rangle\right)$$

Second approach: dynamical higher gradient energy formulation

Insert displacement localizator into Hill-Mandel condition:

$$2.W^{\mathrm{int}}(E,K)=\left(E_{ij}.\Sigma_{ij}+K_{ijk}.S_{ijk}\right)=\left\langle \varepsilon_{ij}\sigma_{ij}\right\rangle+\left\langle \rho\ddot{u}_i\left(u_i-E_{ij}x_j-\frac{1}{2}K_{ijk}x_jx_k\right)\right\rangle$$

$$\Rightarrow \Sigma^{dyn}{}_{ij}=\frac{\partial W^{\mathrm{int}}(E,K,\ddot{E},\ddot{K})}{\partial E_{ij}}+\frac{\partial^2}{\partial t^2}\left(\frac{\partial W^{\mathrm{int}}(E,K,\ddot{E},\ddot{K})}{\partial \ddot{E}_{ij}}\right),\quad S_{ijk}=\frac{\partial W^{\mathrm{int}}(E,K,\ddot{E},\ddot{K})}{\partial K_{ijk}}+\frac{\partial^2}{\partial t^2}\frac{\partial W^{\mathrm{int}}(E,K,\ddot{E},\ddot{K})}{\partial \ddot{K}_{ijk}}$$

$$\Sigma^{dyn}{}_{ij}=E_{dl}\left\langle\frac{1}{2}\left(A^E_{krij}C_{krmn}A^E_{mndl}+A^E_{rsdl}.C_{rsmn}.A^E_{mnij}\right)\right\rangle+K_{rst}\left\langle\frac{1}{2}\left(A^E_{klij}C_{klmn}A^K_{mnrst}+A^K_{klrst}C_{klmn}A^E_{mnij}\right)\right\rangle$$

$$+\ddot{E}_{dl}\left\langle\frac{\rho}{2}\left(H^E_{qdl}H^E_{qij}+H^E_{kij}H^E_{kdl}-H^E_{idl}x_j-H^E_{dij}x_l\right)\right\rangle+\ddot{K}_{rst}\left\langle\frac{\rho}{2}\left(H^K_{qrst}H^E_{qij}+H^K_{krst}H^E_{kij}-H^K_{irst}x_j-\frac{1}{2}H^E_{rij}x_sx_t\right)\right\rangle$$

$$S_{ijk}=\left\langle\frac{1}{2}A^E_{rsdl}C_{rsmn}A^K_{mnijk}+\frac{1}{2}A^K_{rsijk}C_{rsmn}A^E_{mndl}\right\rangle.E_{dl}+\left\langle\frac{1}{2}A^K_{pfijk}C_{pfmn}A^K_{mnrst}+\frac{1}{2}A^K_{dlrst}C_{dlmn}A^K_{mnijk}\right\rangle.K_{rst}$$

$$+\left\langle\frac{\rho}{2}\left(H^E_{qdl}H^K_{qijk}+H^E_{rdl}H^K_{rijk}-\frac{1}{2}H^E_{idl}x_kx_j-H^K_{dijk}x_l\right)\right\rangle.\ddot{E}_{dl}+\left\langle\frac{\rho}{2}\left(H^K_{lrst}H^K_{lijk}+H^K_{drst}H^K_{dijk}-\frac{1}{2}H^K_{rijk}x_sx_t-\frac{1}{2}H^K_{irst}x_jx_k\right)\right\rangle.\ddot{K}_{rst}$$

$$\nabla_{X_1}\left(\Sigma^{dyn}{}_{d1}-\nabla_{X_1}S^{dyn}{}_{d11}\right)=\rho\ddot{U}_d$$ Dynamical equ. of motion

Dynamical second gradient models based on higher gradient energy formulation lead to **dynamical effective stress and hyperstress tensors** contrary to Hamilton principle based formulation.

Effective higher gradient dynamic moduli

Harmonic wave Ansatz:

$$\begin{pmatrix} \ddot{E} \\ \ddot{K} \end{pmatrix} = -w^2 \begin{pmatrix} E \\ K \end{pmatrix}$$

$$\Sigma_{ij}^{dyn} = C_{ijdl}^{\mathrm{hom}} E_{dl} + B_{ijrst}^{\mathrm{hom}} K_{rst}$$
$$S_{ijk}^{dyn} = D_{ijkdl}^{\mathrm{hom}} E_{dl} + A_{ijkrst}^{\mathrm{hom}} K_{rst}$$

Dynamical strain gradient constitutive law with frequency-dependent effective moduli

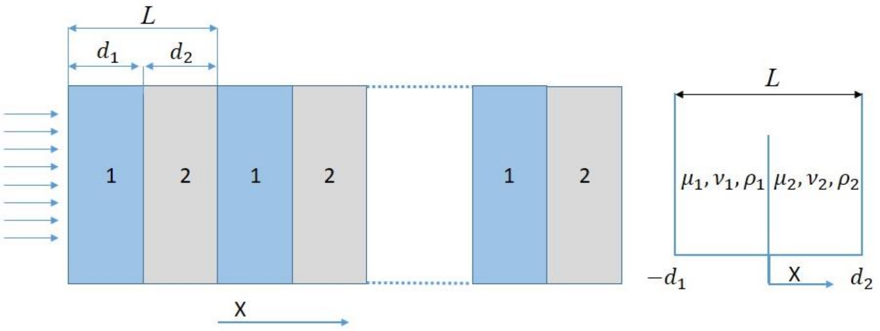
$$C_{ijdl}^{\mathrm{hom},dyn} = \left\langle \frac{1}{2}\left(A_{krij}^{E} C_{krmn} A_{mndl}^{E} + A_{rsdl}^{E} . C_{rsmn} . A_{mnij}^{E} \right) \right\rangle - w^2 \left\langle \frac{\rho}{2} \begin{pmatrix} H_{qdl}^{E} H_{qij}^{E} + H_{kij}^{E} H_{kdl}^{E} \\ -H_{idl}^{E} x_j - H_{dij}^{E} x_l \end{pmatrix} \right\rangle$$

$$B_{ijrst}^{\mathrm{hom},dyn} = \left\langle \frac{1}{2}\left(A_{klij}^{E} C_{klmn} A_{mnrst}^{K} + A_{klrst}^{K} C_{klmn} A_{mnij}^{E} \right) \right\rangle - w^2 \left\langle \frac{\rho}{2} \begin{pmatrix} H_{qrst}^{K} H_{qij}^{E} + H_{krst}^{K} H_{kij}^{E} \\ -H_{irst}^{K} x_j - \frac{1}{2} H_{rij}^{E} x_s x_t \end{pmatrix} \right\rangle$$

$$D_{ijkdl}^{\mathrm{hom},dyn} = \left\langle \frac{1}{2} A_{rsdl}^{E} C_{rsmn} A_{mnijk}^{K} + \frac{1}{2} A_{rsijk}^{K} C_{rsmn} A_{mndl}^{E} \right\rangle - w^2 \left\langle \frac{\rho}{2} \begin{pmatrix} H_{qdl}^{E} H_{qijk}^{K} + H_{rdl}^{E} H_{rijk}^{K} \\ -\frac{1}{2} H_{idl}^{E} x_k x_j - H_{dijk}^{K} x_l \end{pmatrix} \right\rangle = B_{dlijk}^{\mathrm{hom},dyn}$$

$$A_{ijkrst}^{\mathrm{hom},dyn} = \left\langle \frac{1}{2} A_{pfijk}^{K} C_{pfmn} A_{mnrst}^{K} + \frac{1}{2} A_{dlrst}^{K} C_{dlmn} A_{mnijk}^{K} \right\rangle - w^2 \left\langle \frac{\rho}{2} \begin{pmatrix} H_{lrst}^{K} H_{lijk}^{K} + H_{drst}^{K} H_{dijk}^{K} \\ -\frac{1}{2} H_{rijk}^{K} x_s x_t - \frac{1}{2} H_{irst}^{K} x_j x_k \end{pmatrix} \right\rangle$$

Effective strain gradient moduli of stratified lamina



Composite with periodic, layered microstructure

Microscopic displacement for longitudinal motion:

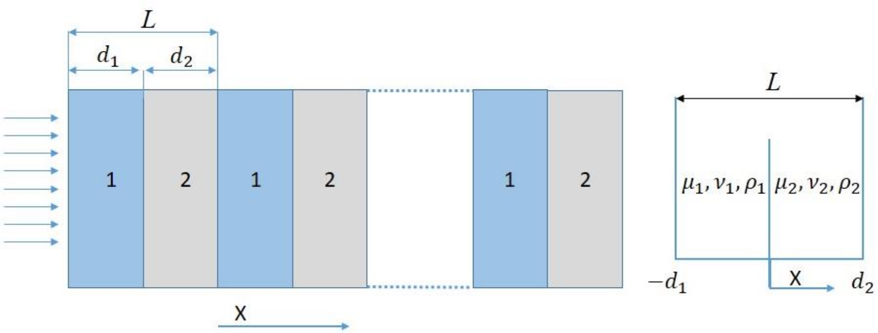
$$\begin{cases} u_1^1(x) = \begin{bmatrix} a_0 + a_1 x & a_2 + a_3 x + a_4 \dfrac{x^2}{2} \end{bmatrix} . \begin{bmatrix} E_{11} \\ K_{111} \end{bmatrix} \\ u_1^2(x) = \begin{bmatrix} b_0 + b_1 x & b_2 + b_3 x + b_4 \dfrac{x^2}{2} \end{bmatrix} . \begin{bmatrix} E_{11} \\ K_{111} \end{bmatrix} \end{cases}$$

Boundary and interface conditions:

$$\begin{cases} u_1^1(0) = u_2^1(0) \\ u_1^1(-d_1) = -E_{11} d_1 + K_{111} \dfrac{d_1^{\,2}}{2}, u_2^1(d_2) = E_{11} d_2 + K_{111} \dfrac{d_2^{\,2}}{2} \\ \sigma_{11}^1(0) = \sigma_{11}^2(0), \dfrac{\partial \sigma_{11}^1}{\partial x}(0) = \dfrac{\partial \sigma_{11}^2}{\partial x}(0) \\ \langle \varepsilon_{11} \rangle_\Omega = E_{11}, \langle \nabla \varepsilon_{11} \rangle_\Omega = K_{111} \end{cases}$$

⟶ Identify coefficients

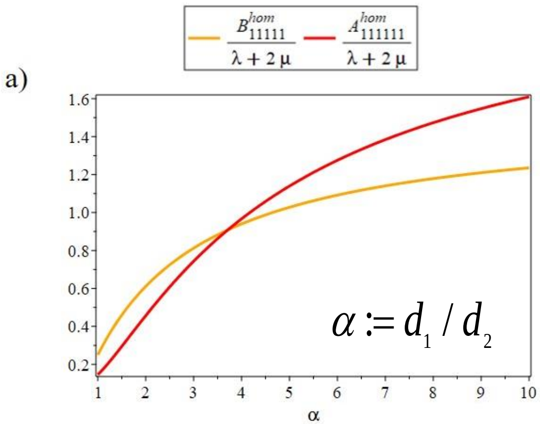
Effective strain gradient moduli of stratified lamina (2)

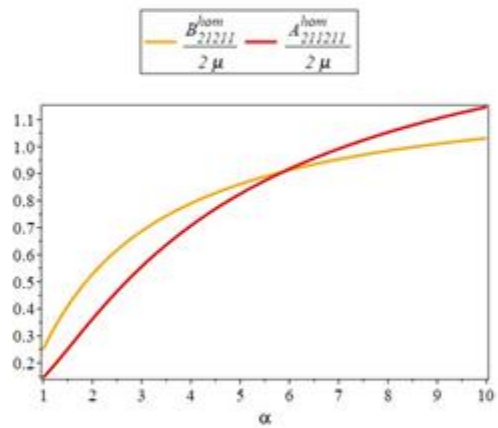


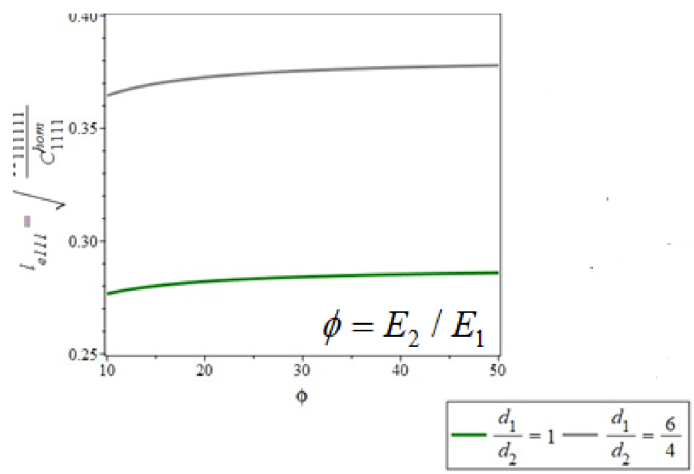
Geometric and mechanical layers properties.

Properties	First layer	Second layer
Thickness	50%	50%
Poisson's ration	$\nu = 0.3$	$\nu = 0.3$
Shear modulus	$\mu_1 = 80760\text{MPa}$	$\mu_2 = 8076\text{MPa}$
Mass density	$\rho_1 = 8000\text{Kg/m}^3$	$\rho_2 = 800\text{Kg/m}^3$

Contrast of shear moduli by a factor 10



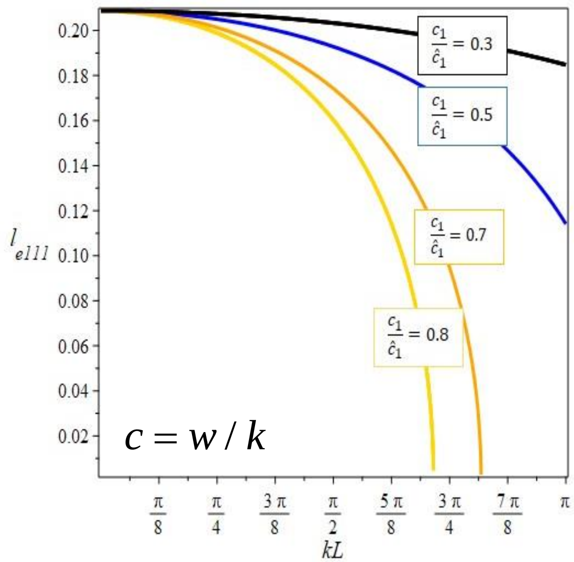




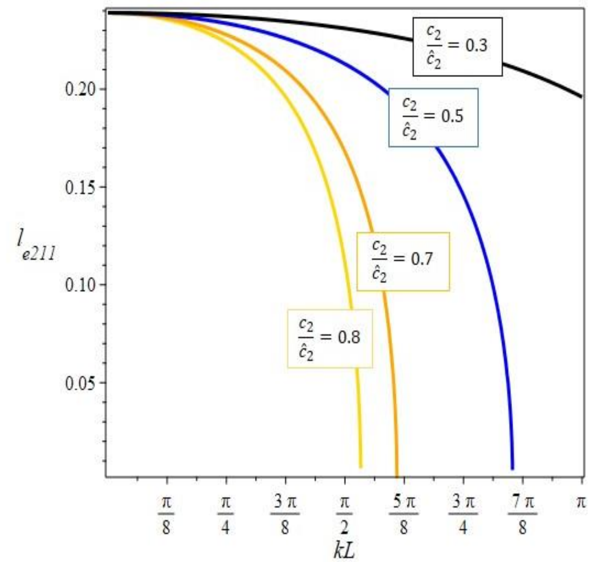
- Increasing geometric discrepancies of the layers yields higher strain gradient contributions
- Most prominent changes due to geometrical contrast effects rather than material contrast

Dynamical characteristic lengths of stratified lamina

Longitudinal waves



Transverse waves



Variation of dynamic second gradient characteristic length versus normalized wavenumber for different values of the phase velocity for longitudinal (left) and transverse (right) motions.

Phase velocity at zero frequency: $\left(\hat{c}_d = \sqrt{C_{d1d1}^{\text{hom},Static} / \rho^{\text{hom}}}, w = 0\right)$

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Shear modulus	$\mu_1 = 80760\text{MPa}$	$\mu_2 = 8076\text{MPa}$
Mass density	$\rho_1 = 8000\text{Kg/m}^3$	$\rho_2 = 800\text{Kg/m}^3$

$\hat{c}_1 = 3.417 \cdot 10^3 m/s, \ \ \hat{c}_2 = 2.58 \cdot 10^3 \text{m/s}$

Wave propagation features of laminated composites using dynamic higher gradient homogenization (DHGH)

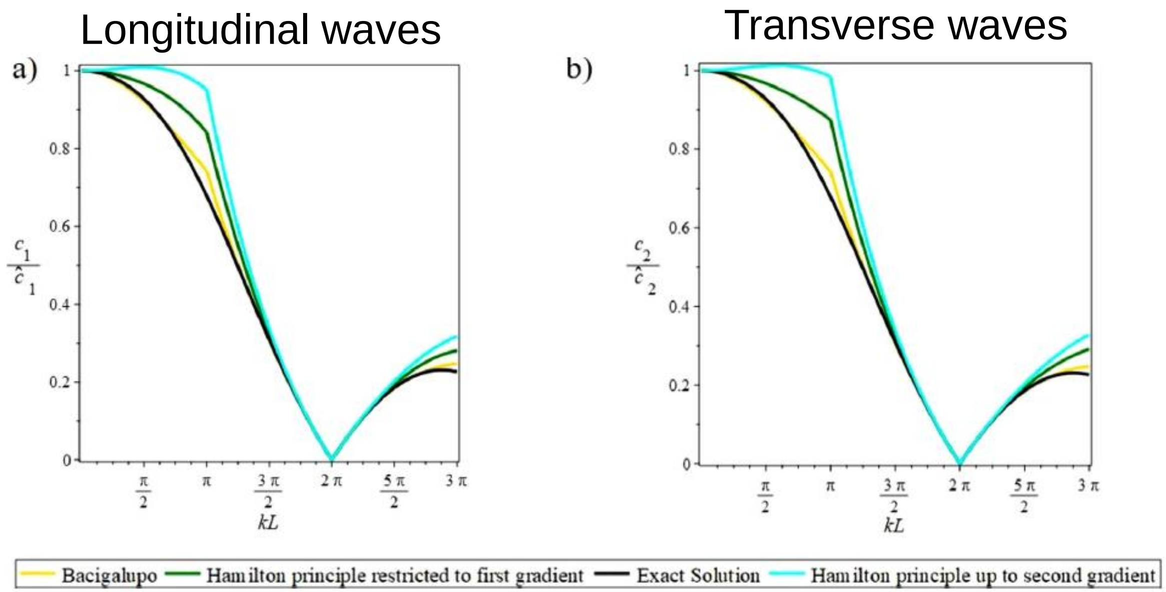
Dispersion relation for longitudinal & transverse modes:

$$\frac{1}{d}\left((C_1^d \text{-} C_2^d \cdot \xi^2 c^2)\xi^2 \cdot (C_7)^3 + (C_5^d \text{-} C_6^d \cdot \xi^2 c^2)\xi^4 C_7\right) \text{-} \rho^{\text{hom}} (C_7)^3 \xi^2 c^2 = 0$$

$$\xi = kL$$

Constants value obtained by the DHGE method.

Mode constants	Longitudinal mode	Transverse mode
C_1^d	$51392.72 MPa$	$58734.54 MPa$
C_2^d	$368.1 kg/m^3$	$1227.27 kg/m^3$
C_3^d	$-2336 MPa \cdot m$	$-2669 MPa \cdot m$
C_4^d	$-152 kg/m^2$	$-289 kg/m^2$
C_5^d	$2238 MPa \cdot m^2$	$2558 MPa \cdot m$
C_6^d	$59\ kg/m$	$78 kg/m$
C_7	$1m$	$1m$
$\hat{c}_d$	$3.417 \cdot 10^3 m/s$	$2.58 \cdot 10^3 m/s$



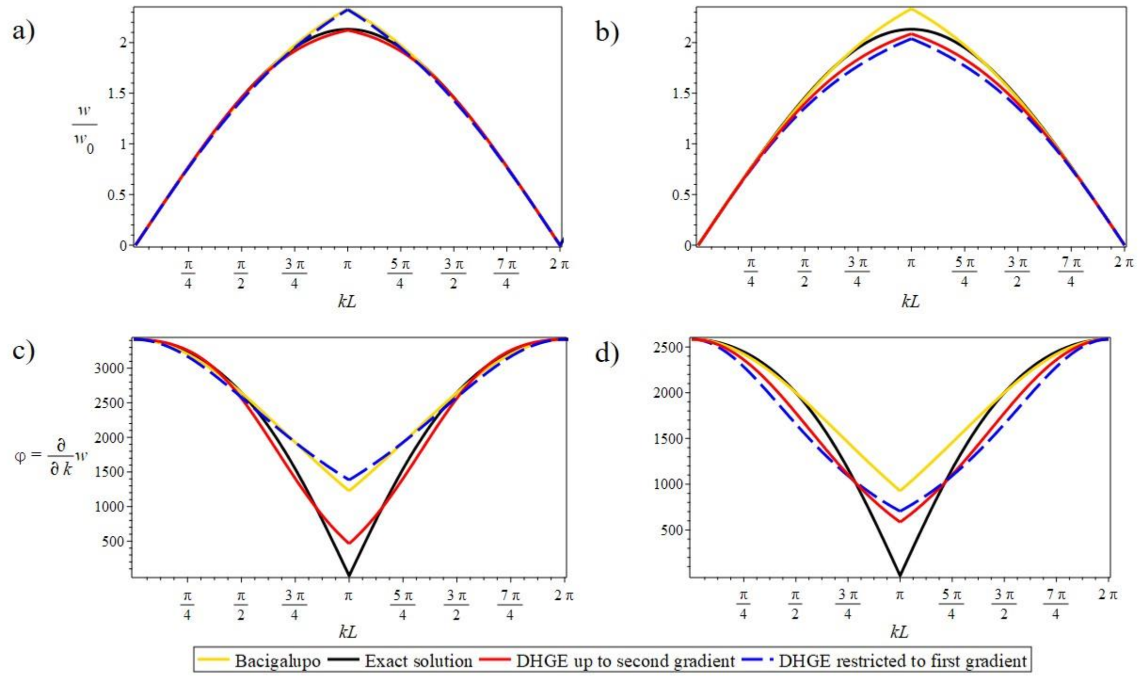
- Very good agreement of DHGH with exact solution (black) over entire range of normalized wave number values both for longitudinal and transverse modes.

Wave propagation features of laminated media using DHGH

Longitudinal waves | Transverse waves

Phase velocity

Group velocity



Frequency of longitudinal (a) and transverse waves (b) and group velocity for longitudinal (c) and transverse waves (d) versus the propagating wavenumber value

- Exact solution (Floquet-Bloch theory) predicts maximum frequency for wavenumber $kL = \pi$ with continuous frequency gradient & nil group velocity - > **standing waves**

- First gradient theory predicts frequencies with jump of slope.

- Second gradient homogenization closer to minimum group velocity compared to other models

- All other approaches lead to more prominent discontinuity point for $kL = \pi$

Vibration features of microstructured beams based on higher gradient dynamic homogenization

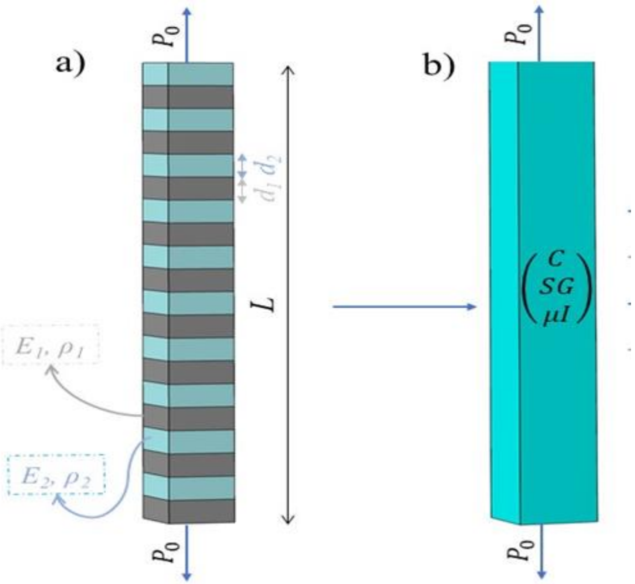
- Higher gradients contributions affect structural vibrations (Altan et al.,1996; Salehian & Inman, 2008; Bisegna & Caruso, 2011) - higher order frequencies more sensitive to gradient effects (Altan et al.,1996).

- No exact 2nd gradient homogenization method to characterize static and dynamic response of micro-structured materials upon a direct correlation of micro and macro-scale parameters.

- Effect of micro-inertia disregarded -> significance of micro-inertia contributions on dynamic characteristics (eigenmode) of structural members remains non-quantified.

- Here, use a **higher-gradient homogenization scheme** -> avoids the need to postulate the form of the non-local interactions, while it accounts for effect of micro-inertia.

Vibration features of microstructured beams based on higher gradient dynamic homogenization (2)



Material	(GPa)	(kg/m^3)	(GPa)	(GPa)
Silicon Carbide	400	3000	346	148
Borosilicate Glass	60	2300	16.6	25
E-Glass	72	2500	23	29.5
Epoxy	3.4	1250	2.48	1.28
Carbon	250	1800	115.5	98.4

$$\Sigma_{ij}^{\text{hom},dyn} = \begin{bmatrix} C_{ijdl}^{\text{hom,stat}} & C_{ijdl}^{\text{hom,dyn}} \end{bmatrix} \begin{bmatrix} E_{dl} \\ \ddot{E}_{dl} \end{bmatrix} + \begin{bmatrix} B_{ijrst}^{\text{hom,stat}} & B_{ijrst}^{\text{hom,dyn}} \end{bmatrix} \begin{bmatrix} K_{rst} \\ \ddot{K}_{rst} \end{bmatrix}$$

$$S_{ijk}^{\text{hom},dyn} = \begin{bmatrix} D_{ijkdl}^{\text{hom,stat}} & D_{ijkdl}^{\text{hom,dyn}} \end{bmatrix} \begin{bmatrix} E_{dl} \\ \ddot{E}_{dl} \end{bmatrix} + \begin{bmatrix} A_{ijkrst}^{\text{hom,stat}} \end{bmatrix} \begin{bmatrix} K_{rst} \end{bmatrix}$$

Higher gradient dynamical constitutive law

$$\nabla_X \left(\Sigma_{11} - \nabla_X S_{111} \right) = \rho^{\text{hom}} \frac{\partial^2 U}{\partial t^2}$$

→

$$C_1^1 \frac{\partial^2 U}{\partial X^2} + C_2^1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 U}{\partial X^2} \right) - C_5^1 \frac{\partial^4 U}{\partial X^4} = \rho^{\text{hom}} \frac{\partial^2 U}{\partial t^2}$$ longitudinal dynamical equ.

C_2^1 micro-inertia, $l_e = \sqrt{C_5^1 / C_1^1}$ longitudinal internal length

Vibration features of microstructured beams based on higher gradient dynamic homogenization (3)

Parameters	Cauchy behavior	Second gradient with micro-inertia
Eigenfrequency	$\omega^2 = k^2 \cdot \left(\frac{C_1^1}{\rho^{\text{hom}}}\right)$	$\omega^2 = k^2 \cdot \left(\frac{C_5^1 \cdot k^2 + C_1^1}{k^2 \cdot C_2^1 + \rho^{\text{hom}}}\right)$
Characteristics equation	$\sin(kL) = 0$	$\mathrm{F}(k, \mathrm{L}, C_1^1, C_2^1, C_5^1, \rho^{\text{hom}}) = 0$

Micro-inertia effects on higher gradient dynamic attributes

Eigenfrequency values of the composite Silicon Carbide-Borosilicate Glass beam structure

Mode	Cauchy		SG			SG+μI		
	ξ	Ω_c	ξ	Ω_{SG}	$\%(\Omega_{SG}-\Omega_c)/\Omega_c$	ξ	$\Omega_{SG+\mu I}$	$\%(\Omega_{SG+\mu I}-\Omega_c)/\Omega_c$
1	π	3.14	3.28	3.28	4.6	3.28	3.27	4.1
2	2π	6.28	6.55	6.61	5.3	6.55	6.53	4
3	3π	9.42	9.82	10.01	6.3	9.83	9.75	3.5

- Significant frequency difference between Cauchy predictions and strain gradient with microinertia
- Microinertia terms counteract strain gradient terms - important even for low internal lengths

Summary - Outlook

- Important effect of micro-inertia on eigenfrequencies.

- High accurate predictions of dispersion relations over entire wavenumber space compared to exact Foquet-Bloch predictions.

- Microinertia effects on vibrations counteract strain gradient effects

- Micromorphic dynamical models formulated by homogenization

Int. J. Engng Sci., 2020 -> eigenfrequencies of composite beams using higher-order homogeizization.
Int. J. Solids Struct., 2019 -> statics and dynamics of heterogeneous media with microinertia