Unfolding of a bistable tape spring: analogy with a regularized bistable Ericksen bar

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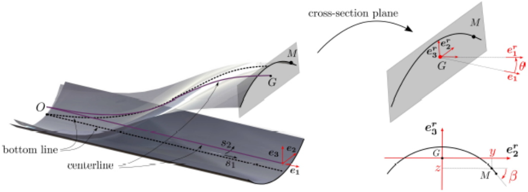
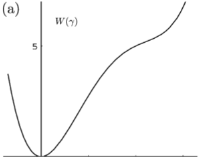
Dynamics of a bistable tape spring

- long fiber composite tape, with 2 stable equilibrium positions:
  - ✓ unrolled (curved cross section)
  - ✓ coiled (flat cross section)
- slow variation of the boundary conditions:
  - ✓ fast deployment of the tape
  - ✓ 3 zones: unfolded / transition area / coiled up

Models

- many models: 3D, shells, flexible section beams
  - Picault-Bourgeois-Cochelin-Guinot (2016): 7 dof (3D)
  - Guinot-Bourgeois-Cochelin-Blanchard (2012): 4 dof (tension, plane flexion)
  - Martin-Bourgeois-Cochelin-Guinot (2020): 2 dof (pure flexion θ and β)





- statics: analogy with a regularized Ericksen bar

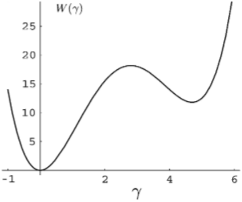
  longitudinal curvature $\mapsto$ strain

  ✓ local non-convexity of energy: formation of folds

  ✓ regularization: transition zones, finite number of folds

  ✓ higher-order boundary conditions: variation of the cross-section shape

Objective of the study



- a step further for the analogy tape spring / Ericksen bar
  - ✓ introduction of dynamics and bistability
  - ✓ elementary model: 1 dof
- capture the main features observed
  - ✓ switching from one stable state to another via a boundary condition
  - ✓ propagation of a transition zone at constant speed (travelling wave)

Sketch of the study

1. bistable regularized Ericksen bar

   Lagrangian, PDE, exact solution (kink wave)

2. augmented Lagrangian

   PDE, boundary conditions, hyperbolic system

3. numerical experiments

   variable boundary conditions, properties of the front

4. conclusion and prospects

Part I

Bistable regularized Ericksen bar

Original Lagrangian

- displacement u, velocity $v = \partial_t u$, strain $\varepsilon = \partial_x u$

  kinetic energy $\mathcal{T}$, potential energy $\mathcal{V}$, Lagrangian $\mathcal{L}$

$$\mathcal{T} = \frac{\rho}{2}(\partial_t u)^2, \qquad \mathcal{V} = W(\varepsilon) + \frac{\alpha}{2}(\partial_x \varepsilon)^2, \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

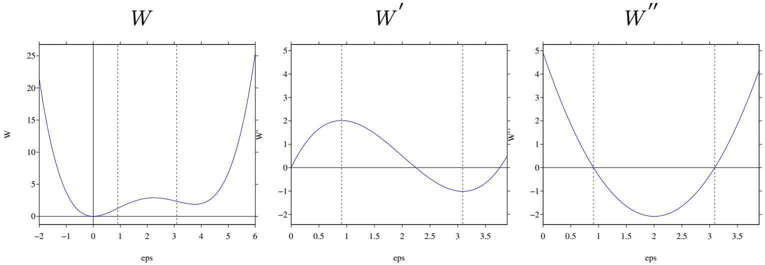
- Ericksen energy: W' "up-down-up"

$$W(\varepsilon) = a_2\,\varepsilon^2 + a_3\,\varepsilon^3 + a_4\,\varepsilon^4$$

- properties:

  ✓ W concave on $]\varepsilon_1, \varepsilon_2[$

  ✓ $W' = 0$ at 3 equilibrium points: 0 (stable) $< \varepsilon_1^0$ (unstable) $< \varepsilon_2^0$ (stable)



Evolution equations

- Euler-Lagrange

$$\begin{cases} \partial_t \varepsilon - \partial_x v = 0 \\ \rho\,\partial_t v - \partial_x \sigma = 0, \qquad \sigma = W'(\varepsilon) - \alpha\,\partial^2_{xx}\varepsilon \end{cases}$$

- energy balance: $\dfrac{d}{dt}\mathcal{E} = \mathcal{P}$

$$\mathcal{E} = \int_0^L \left(\frac{\rho}{2} v^2 + W(\varepsilon) + \frac{\alpha}{2}(\partial_x \varepsilon)^2 \right) dx, \qquad \mathcal{P} = \left[v\,\sigma + \alpha\,\partial_t \varepsilon\,\partial_x \varepsilon \right]_0^L$$

- admissible boundary conditions:

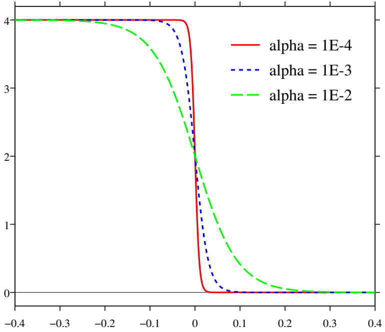
$$\begin{cases} v(0,t) = 0 \\ \partial_x \varepsilon(0,t) = 0 \\ \varepsilon(L,t) = f_\varepsilon(t) \\ \sigma(L,t) = 0 \end{cases}$$

✓ steady forcing $f_\varepsilon \Rightarrow \dfrac{d}{dt}\mathcal{E} = 0$

Exact solution

- dimensional analysis $V \propto \sqrt{\frac{a_i}{\rho}}$, $D \propto \sqrt{\frac{\alpha}{a_i}}$
- travelling wave $\varepsilon(\xi)$, with $\xi = x - Vt$
- kink wave between 2 states $\varepsilon_A = 0 < \varepsilon_B = -\dfrac{a_3}{2\,a_4}$

$$V = \sqrt{\frac{2}{\rho}\left(a_2 - \frac{a_3^2}{4\,a_4}\right)}, \qquad \varepsilon(\xi) = \frac{\varepsilon_B}{1 + \exp\left(\varepsilon_B \sqrt{\frac{2\,a_4}{\alpha}}\,\xi\right)}$$



Dispersive system

- 1^{st} order in time

$$\partial_t \mathbf{U} + \partial_x \mathbf{f}(\mathbf{U}) = \mathbf{S}\, \partial^3_{xxx}$$

$$\mathbf{U} = \begin{pmatrix} \varepsilon \\ v \end{pmatrix}, \qquad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} -v \\ -\dfrac{1}{\rho} W'(\varepsilon) \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$$

- eigenvalues of the Jacobian $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{U}}$

$$\Lambda^{(1,2)} = \pm \sqrt{\frac{W''(\varepsilon)}{\rho}}$$

- $\varepsilon \in]\varepsilon_1, \varepsilon_2[\Rightarrow W''(\varepsilon) < 0$: imaginary sound speed

✗ not well-posed on $[\varepsilon_1, \varepsilon_2]$

✗ inversion of an elliptic operator at each time step

Part II

Augmented Lagrangian for a bistable Ericksen bar

Augmented Lagrangian

- original Lagrangian $\mathcal{L}$ (reminder): kinetic energy $\mathcal{T}$, potential energy $\mathcal{V}$

$$\mathcal{T} = \frac{\rho}{2}\,(\partial_t u)^2, \qquad \mathcal{V} = W(\varepsilon) + \frac{\alpha}{2}\,(\partial_x \varepsilon)^2, \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

- augmented Lagrangian $\mathcal{L}_e$: variable η, kinetic energy $\mathcal{T}_e$, potential energy $\mathcal{V}_e$

$$\mathcal{T}_e = \frac{\rho}{2}\,(\partial_t u)^2 + \frac{\beta}{2}\,(\partial_t \eta)^2, \qquad \mathcal{V}_e = W(\varepsilon) + \frac{\alpha}{2}\,(\partial_x \eta)^2 + \frac{\lambda}{2}\,(\varepsilon - \eta)^2, \qquad \mathcal{L}_e = \mathcal{T}_e - \mathcal{V}_e$$

- parameters:

  micro-inertia β $\curvearrowright$ equation for η (e.g. : Poisson effect)

  penalization λ $\curvearrowright$ real sound speed (hyperbolicity)

- property (to be proven):

  ✓ $\eta = \varepsilon + \mathcal{O}(\lambda^{-1}) + \mathcal{O}(\beta^n),\ n \geq 0$

Evolution equations

- Euler-Lagrange: $w = \partial_t \eta$, $p = \partial_x \eta$

$$\begin{cases} \partial_t \varepsilon - \partial_x v = 0 \\ \rho\, \partial_t v - \partial_x \sigma_e = 0, \qquad \sigma_e = W'(\varepsilon) + \lambda\, (\varepsilon - \eta) \\ \partial_t \eta = w \\ \partial_t p - \partial_x w = 0 \\ \beta\, \partial_t w - \partial_x (\alpha\, p) = \lambda(\varepsilon - \eta) \end{cases}$$

- energy balance: $\dfrac{d}{dt} \mathcal{E}_e = \mathcal{P}_e$

$$\mathcal{E}_e = \int_0^L \left(\frac{\rho}{2} v^2 + \frac{\beta}{2} w^2 + W(\varepsilon) + \frac{\alpha}{2} p^2 + \frac{\lambda}{2} (\varepsilon - \eta)^2 \right) dx, \quad \mathcal{P}_e = [v\, \sigma_e + \alpha\, p\, w]_0^L$$

- admissible boundary conditions:

$$\begin{cases} v(0,t) = 0 \\ p(0,t) = 0 \\ \eta(L,t) = f_\eta(t) \\ \varepsilon(L,t) = s^{-1}(\lambda f_\eta(t)) \equiv f_\varepsilon(t) \end{cases}$$

✓ steady forcing $f_\eta \Rightarrow \dfrac{d}{dt} \mathcal{E}_e = 0$

Hyperbolic system

- 1^{st} order in space and time

$$\partial_t \mathbf{U}_e + \partial_x \mathbf{f}_e(\mathbf{U}_e) = \mathbf{S}_e \, \mathbf{U}_e$$

$$\mathbf{U}_e = (\varepsilon, v, \eta, p, w)^\top, \quad \mathbf{f}_e(\mathbf{U}_e) = \left(-v, -\frac{1}{\rho}\sigma_e(\varepsilon,\eta), 0, -w, -\frac{\alpha}{\beta}\, p \right)^\top$$

- sound speed: eigenvalues $\Lambda_e^{(i)}$ of the Jacobian $\mathbf{A}_e = \frac{\partial \mathbf{f}_e}{\partial \mathbf{U}_e}$

$$\Lambda_e^{(1)} = 0, \quad \Lambda_e^{(2,3)} = \pm\sqrt{\frac{W''(\varepsilon) + \lambda}{\rho}}, \qquad \Lambda_e^{(4,5)} = \pm\sqrt{\frac{\alpha}{\beta}}$$

$$\varepsilon \in [\varepsilon_1, \varepsilon_2] \;\Rightarrow\; 0 \le W''(\varepsilon) \ge W''_{\min} \equiv -25/12$$

$\lambda > \lambda_{\min} \equiv 25/12 \Rightarrow$ unconditional hyperbolicity

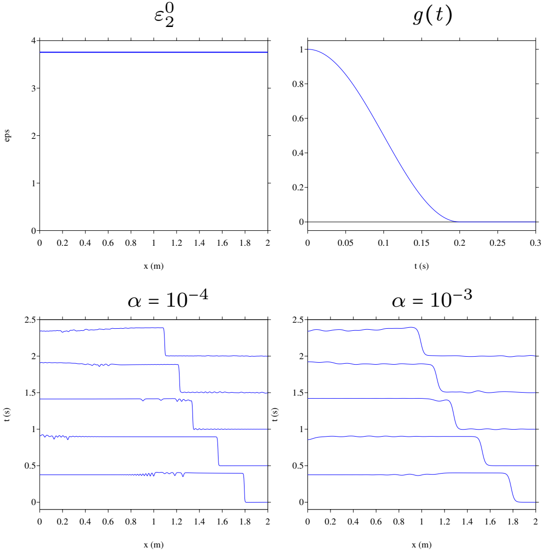
- standard numerical methods for hyperbolic systems with relaxation

Part III

Numerical experiments

Variable boundary conditions

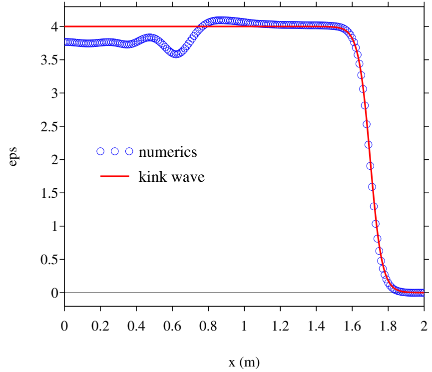
- ✓ $\eta(L,t) = \varepsilon_2^0\, g(t)$ with g decreasing quasi-statically
- ✓ propagation of a front: D increase with α



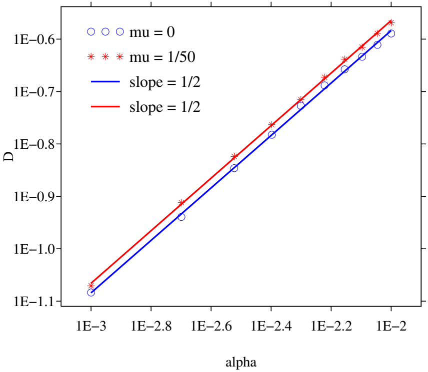
Properties

- ✓ simulation (augmented Lagrangian) vs kink wave (original Lagrangian)
- ✓ velocity (measured and exact): $V = 0.5$ m.s^{-1} depends only on the energy
- ✓ dimensional analysis and kink wave: $D \propto \sqrt{\alpha}$

snapshot de ε



D vs α



Influence of dissipation

- original Lagrangian: Kelvin-Voigt, μ dynamic viscosity

$$\rho\,\partial_t v - \partial_x \sigma = \mu\,\partial^2_{xx} v$$

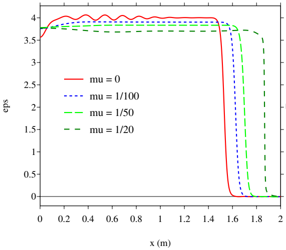
- augmented Lagrangian: relaxation term

$$\beta\,\partial_t w - \partial_x(\alpha\,p) = \lambda(\varepsilon - \eta) - \mu\,w$$

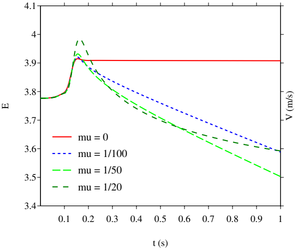
- dimensional analysis

$$V = \Phi_V\left(\sqrt{\frac{a_i}{\rho}},\ \frac{\sqrt{a_i\,\alpha}}{\mu}\right)$$

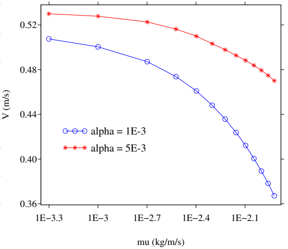
snapshot of ε



energy dissipation



V vs μ



Part IV

Conclusion and prospects

In brief

- regularized bistable Ericksen bar (1 dof): "up-down-up" energy $+ \alpha\partial^2_{xx}u$
  - ✓ existence of kink waves
  - ✓ boundary conditions: switch between two stable equilibria
  - ✓ efficient numerical strategy (augmented Lagrangian)
  - Bourgeois-Favrie-Lombard, IJSS (2020), to appear
- future directions:
  - ✓ mathematical analysis (Duchêne, Nonlinearity 2019)
  - ✓ 2D and finite-strain
  - ✓ design of innovative deployable structures (nonlinear metamaterials)

Thanks for your attention!

Part V

Appendix

Original Lagrangian: dispersion analysis

- linearization $\varepsilon = \overline{\varepsilon} + \zeta\,\tilde{\varepsilon}$, with $\tilde{\varepsilon} = \hat{\varepsilon}\,e^{i(\omega t - kx)}$ and $0 < \zeta \ll 1$: dispersion relation

$$\omega(k) = k\sqrt{\frac{W'' + \alpha\,k^2}{\rho}}, \qquad c_p(k) = \frac{\omega(k)}{k} = \sqrt{\frac{W'' + \alpha\,k^2}{\rho}}$$

- $\overline{\varepsilon} \notin]\varepsilon_1, \varepsilon_2[$: $W'' > 0$

  ✓ $\forall k$: propagation

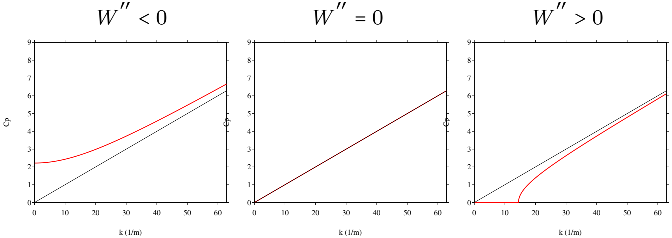
- $\overline{\varepsilon} \in [\varepsilon_1, \varepsilon_2]$: $W'' \leq 0 \Rightarrow$ critical wavenumber $k_c = \sqrt{-W''/\alpha}$

  ✓ $k > k_c$: propagation

  ✗ $k < k_c$: stabilisation with 2 possibilities

  (i) exponential growth of $\tilde{\varepsilon} \curvearrowright \varepsilon \notin [\varepsilon_1, \varepsilon_2]$

  (ii) increase of $k \curvearrowright k > k_c$



Approach: augmented Lagrangian method

- principle: introduction of penalized auxiliary variables in the Lagrangian
  - ✓ real sound speed (unconditional hyperbolicity)
  - ✓ large penalization: augmented Lagrangian → original Lagrangian
  - ✓ similar dispersion properties
- recent works :
  - Favrie-Gavrilyuk (Nonlinearity 2017): Serre-Green-Naghdi
  - Dhaouadi-Favrie-Gavrilyuk (SAP 2019): non-linear Schrödinger
  - Duchêne (Nonlinearity 2019): mathematical analysis

Augmented Lagrangian: toy-model

- original Lagrangian: $\mathcal{L} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{x^2}{2}$

  Euler-Lagrange

$$\begin{cases} \dfrac{d^2x}{dt^2} + x = 0 \\ x(0) = A, \quad \dfrac{dx}{dt}(0) = B \end{cases} \Rightarrow x(t) = A\cos t + B\sin t$$

- Augmented Lagrangian : $\mathcal{L}_e = \frac{1}{2}\left(\frac{dy_e}{dt}\right)^2 - \frac{x_e^2}{2} - \frac{\lambda}{2}(y_e - x_e)^2$

  Euler-Lagrange

$$\begin{cases} \dfrac{d^2y_e}{dt^2} + \omega^2 y_e = 0, \quad x_e = \omega^2 y_e \\ x_e(0) = A, \quad \dfrac{dx_e}{dt}(0) = B \\ \omega^2 = \dfrac{\lambda}{\lambda + 1} \end{cases} \Rightarrow x_e(t) = A\cos\omega t + \frac{B}{\omega}\sin\omega t$$

- error of model: $|x(t) - x_e(t, \lambda)| = \mathcal{O}\left(\frac{1}{\lambda}\right)$

Augmented Lagrangian: dispersion analysis

- quartic equation: 4 solutions $\pm\omega^{\pm}$

$$A_4\,\omega^4 + B(k,\varepsilon)\,\omega^2 + C(k,\omega) = 0$$

$$c^{\pm}(k) = \frac{\omega^{\pm}(k)}{k} \;:\; c_p^+ \text{ fast wave, } c_p^- \text{ slow wave}$$

- properties:

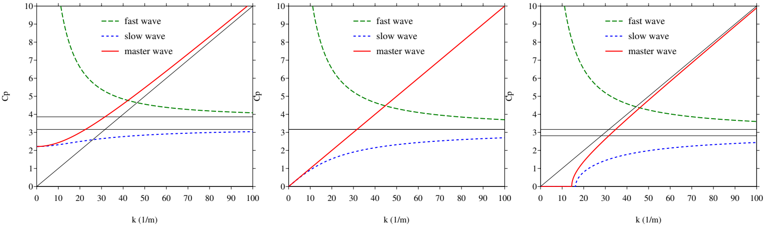
✓ fast wave ≡ spurious wave (acoustic)

$$c_p^+ \in \mathbb{R}^+ \quad \forall k > 0, \quad c_p^+(k) \underset{0}{\sim} \frac{1}{k} \to +\infty$$

✓ slow wave ≡ physical wave

$$c_p^- \in \mathbb{R}^+ \quad \text{si} \left(W'' > 0 \text{ ou } k > k_c^e = \sqrt{-\frac{W''\lambda}{\alpha(W''+\lambda)}} \right), \text{ autrement } c_p^- \in i\,\mathbb{R}$$

$W'' < 0$ | $W'' = 0$ | $W'' > 0$



Numerical methods

- nonlinear hyperbolic system with linear source term

$$\boxed{\frac{\partial}{\partial t}\mathbf{U}_e + \frac{\partial}{\partial x}\mathbf{f}_e(\mathbf{U}_e) = \mathbf{S}_e\,\mathbf{U}_e}$$

standard approach: Strang splitting (propagation and relaxation)

$$\frac{\partial}{\partial t}\mathbf{U}_e + \frac{\partial}{\partial x}\mathbf{f}_e(\mathbf{U}_e) = \mathbf{0} \qquad (\mathbf{H}_p)$$

$$\frac{\partial}{\partial t}\mathbf{U}_e = \mathbf{S}_e\,\mathbf{U}_e \qquad (\mathbf{H}_r)$$

- propagation step: finite-volume scheme with flux limiters

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}\right), \qquad \Delta t \leq \frac{\Delta x}{c_{\max}}$$

boundary conditions: ghost cells (LeVeque 2002)

$$\varepsilon(0,t) = 0 \quad \rightarrow \quad \varepsilon^\star_{-j} = +\varepsilon^n_j$$

$$v(0,t) = f_v(t) \quad \rightarrow \quad v^\star_{-j} = 2\,f_v(t_n) - v^n_j$$

- relaxation setp : exact solution

$$\mathbf{H}_r\,(\tau)\;\mathbf{U}_i = \exp\left(\mathbf{S}_e\tau\right)\mathbf{U}_i$$

Choice of parameters $\beta \to 0$ and $\lambda \to +\infty$

✓ hyperbolicity: $\lambda \geq \lambda_{\min} \equiv 25/12$

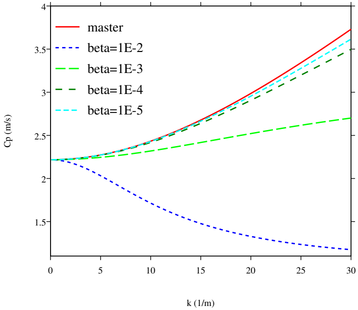
✓ $W'' \geq 0$: increasing slow wave → master wave if $\frac{\alpha}{\beta} > \frac{W''(\varepsilon)}{\rho}$

$$\varepsilon \in [0,\, \varepsilon_2^0] \;\Rightarrow\; \Gamma = \frac{\rho}{2\, a_2}\, \frac{\alpha}{\beta} \geq 1$$

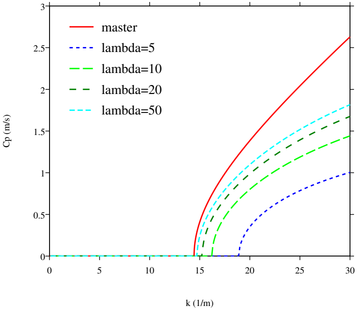
✓ $W'' < 0$: critical wavenumber $\left| \frac{k_c^e}{k_c} - 1 \right| \leq \frac{1}{2\,\lambda} \left| W''_{\min} \right| \equiv \frac{25}{24}\, \frac{1}{\lambda}$

✓ minimization of the numerical dissipation: $\Lambda_e^{(2)}(\varepsilon) = \Lambda_e^{(4)} \;\Rightarrow\; \lambda + 2\, a_2 = \rho\, \frac{\alpha}{\beta}$

$W'' > 0,\ \lambda = 100$

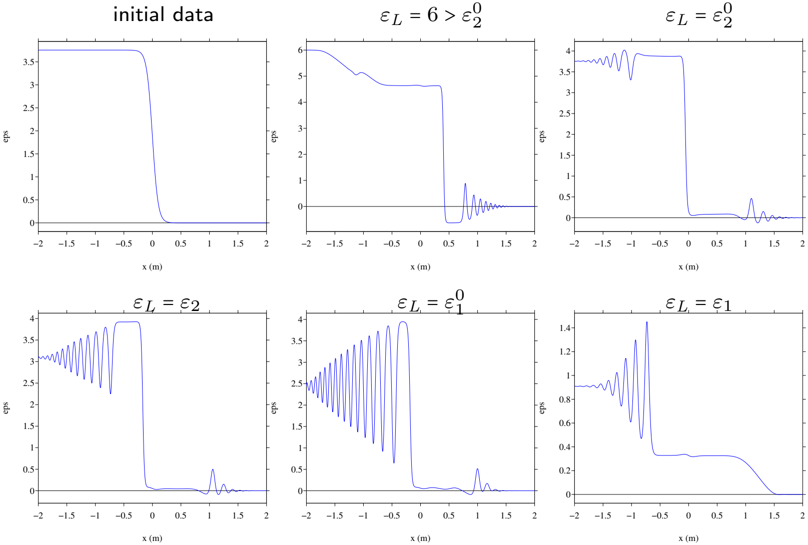


$W'' < 0,\ \beta = 10^{-3}$



Riemann problem

✓ discontinuous initial data $(\varepsilon_L, 0)$, with smooth transition



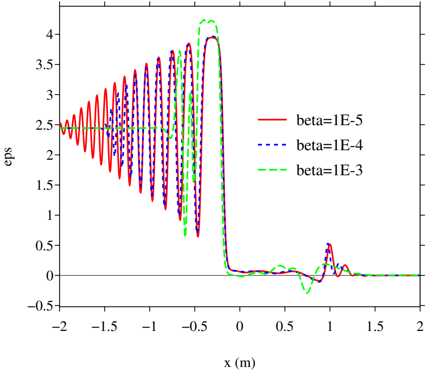
Choice of parameters β and λ

- reference solution: no influence of meshing or (β, λ)
- 2 critèria: $\Gamma(\beta) > 1$, $\lambda > \lambda_c$

✗ criteria not satisfied ($\beta = 10^{-3}$ left, $\lambda = 5$ right) $\rightsquigarrow$ non-convergence

✓ criteria satisfied (other curves) $\rightsquigarrow$ convergence towards the reference solution

β variable et $\lambda = 100$



λ variable et $\beta = 10^{-5}$