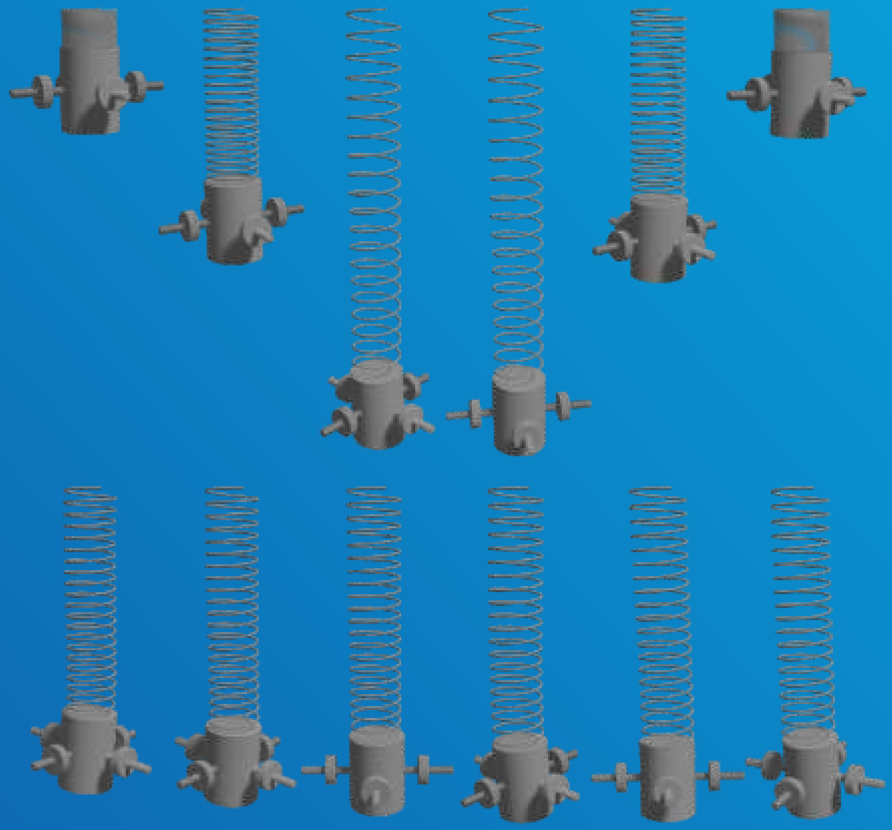
University of Stuttgart
Institute for Nonlinear Mechanics

Advances in ELAstoDYNamics of architeted materials and BIOmaterials



Finite element formulations for constrained spatial nonlinear beam theories*

*submitted to Math. Mech. Solids

Jonas Harsch, Giuseppe Capobianco, Simon R. Eugster

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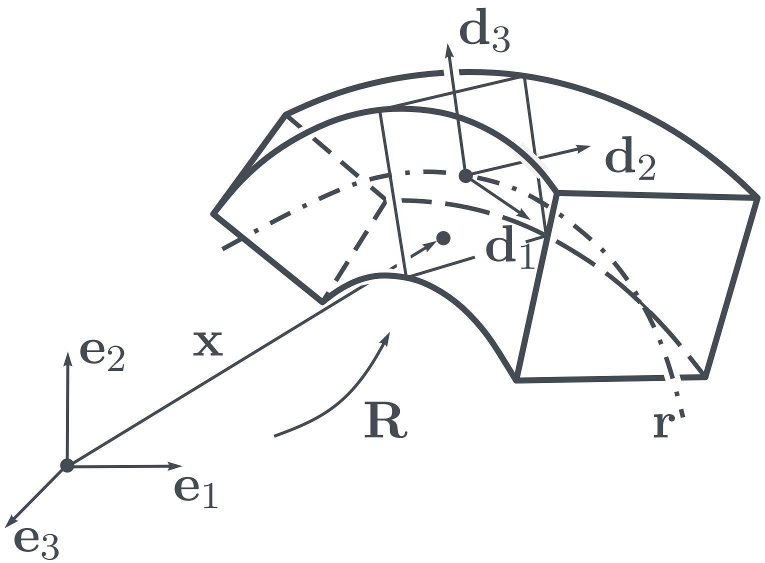
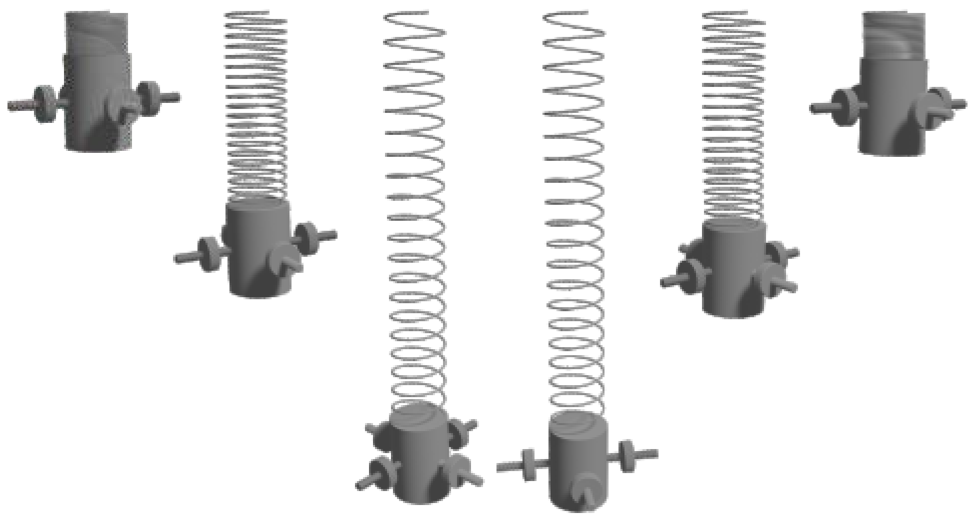
Goal

single finite element formulation for dynamic simulations of classical nonlinear beams

- shear-deformable, shear-rigid and inextensible shear-rigid beams
- large displacements and finite strains
- hyperelastic constitutive laws
- precurved reference configurations





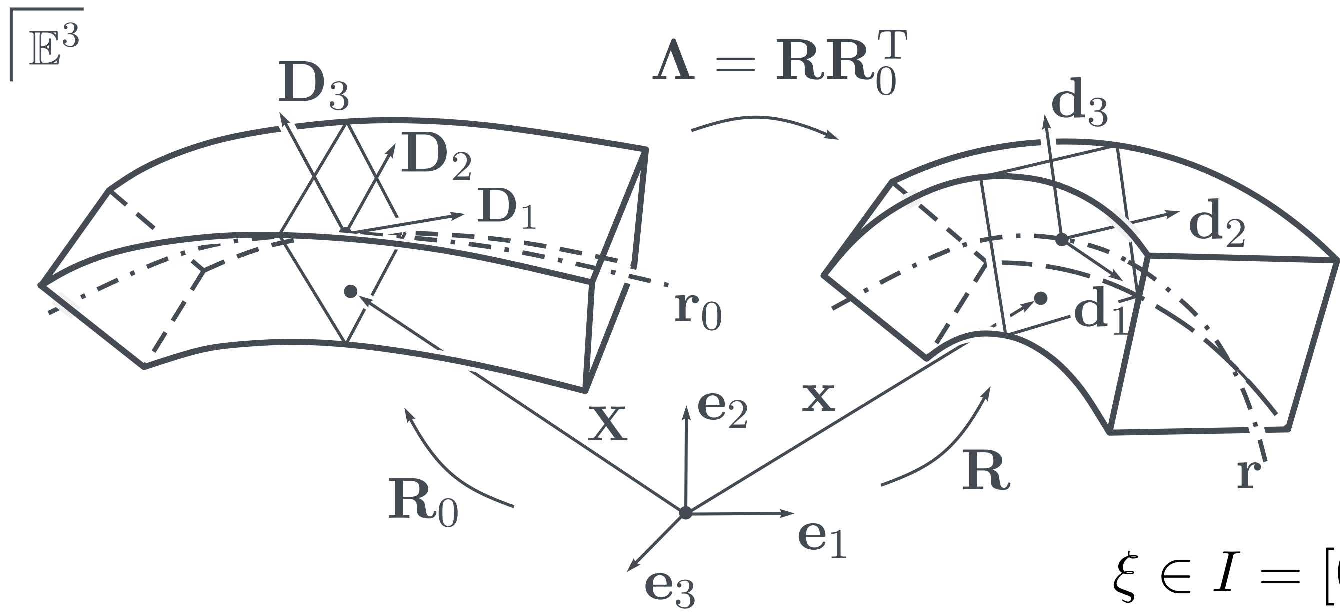
Idea

beam theories as constrained theories

in which

constraints can be switched on and off

Kinematics of shear-deformable beam



$\mathbf{R}, \mathbf{R}_0, \boldsymbol{\Lambda} \in \mathrm{Orth}^+$

$\xi \in I = [0, 1]$ parameter space

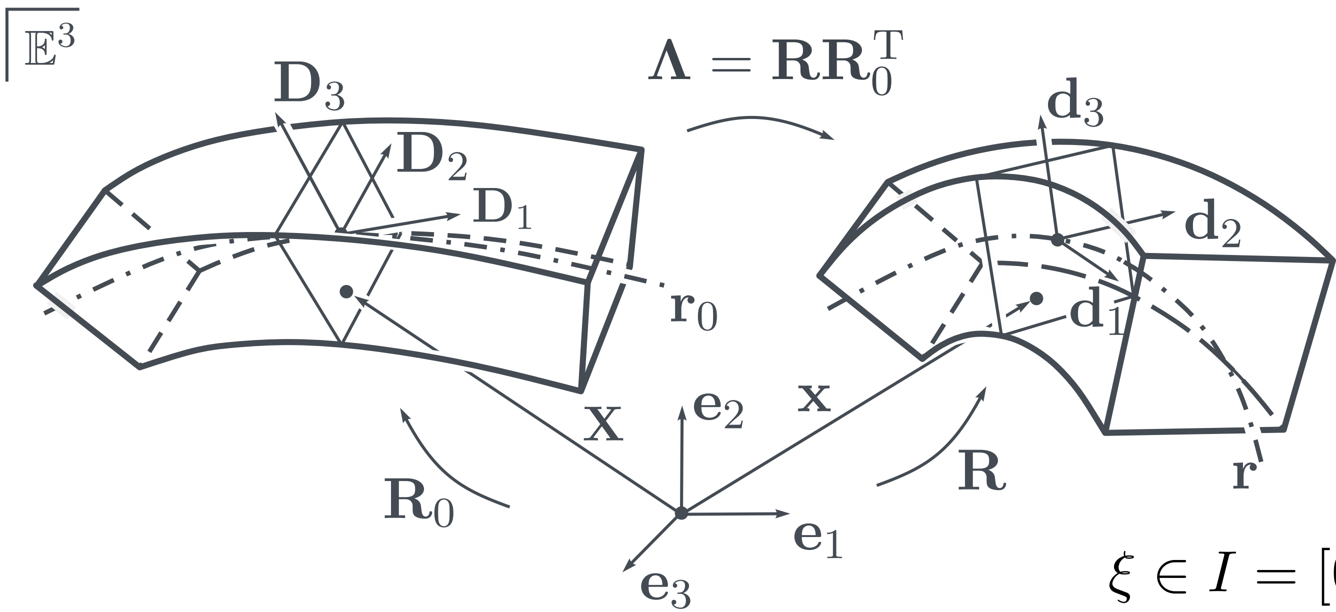
	reference configuration	current configuration
centerline	$\mathbf{r}_0(\xi)$	$\mathbf{r}(\xi, t)$
directors	$\mathbf{D}_i(\xi) = \mathbf{R}_0(\xi)\mathbf{e}_i$	$\mathbf{d}_i(\xi, t) = \mathbf{R}(\xi, t)\mathbf{e}_i$

reference arc-length

$$s = \int_0^{\xi} \underbrace{\|\mathbf{r}_0'(\bar{\xi})\|}_{=:\, J(\bar{\xi})} \mathrm{d}\bar{\xi}$$

$$\frac{\partial f}{\partial s}(\xi, t) := \frac{\partial f}{\partial \xi}(\xi, t)\frac{1}{J(\xi)} = \frac{f'(\xi, t)}{J(\xi)}$$

Objective strain measures



$\mathbf{R}, \mathbf{R}_0, \boldsymbol{\Lambda} \in \mathrm{Orth}^+$

$\xi \in I = [0, 1]$ parameter space

$$\Gamma_i = \frac{1}{J}\, \mathbf{r}' \cdot \mathbf{d}_i$$

$$K_i = \frac{1}{2J} \varepsilon_{ijk} \mathbf{d}_k \cdot \mathbf{d}_j' \,, \;\; K_i^0 = \frac{1}{2J} \varepsilon_{ijk} \mathbf{D}_k \cdot \mathbf{D}_j'$$

$$\boldsymbol{\Gamma} = \Gamma_i \mathbf{D}_i = \boldsymbol{\Lambda}^{\mathrm{T}} \frac{\partial \mathbf{r}}{\partial s}$$

$$\mathbf{K} - \mathbf{K}_0 = (K_i - K_i^0)\mathbf{D}_i = \mathrm{ax}\left(\boldsymbol{\Lambda}^{\mathrm{T}} \frac{\partial \boldsymbol{\Lambda}}{\partial s}\right)$$

- Γ_1 dilatation
- Γ_2, Γ_3 shear
- $\lambda = \|\boldsymbol{\Gamma}\|$ stretch

- $K_1 - K_1^0$ torsion
- $K_2 - K_2^0, K_3 - K_3^0$ flexure

Internal virtual work

total strain energy stored in the beam

$$E = \int_I U(\xi, t) J(\xi)\, \mathrm{d}\xi = \int_I W\big(\Gamma_i(\xi, t), K_i(\xi, t); \xi\big) J(\xi)\, \mathrm{d}\xi$$

$$W(\Gamma_i, K_i; \xi) = \tfrac{1}{2} k_\mathrm{t} \big(K_1 - K_1^0\big)^2 + \tfrac{1}{2} k_{\mathrm{b}_2} \big(K_2 - K_2^0\big)^2 + \tfrac{1}{2} k_{\mathrm{b}_3} \big(K_3 - K_3^0\big)^2 + \\ + \tfrac{1}{2} k_\mathrm{e} \left(\lambda(\Gamma_i) - 1\right)^2 + \tfrac{1}{2} k_{\mathrm{s}_2} \Gamma_2^2 + \tfrac{1}{2} k_{\mathrm{s}_3} \Gamma_3^2$$

internal virtual work

$$\delta W^\mathrm{int} = -\delta E = -\int_I \Big\{ \underbrace{\frac{\partial W}{\partial \Gamma_i}}_{n_i} \delta \Gamma_i + \underbrace{\frac{\partial W}{\partial K_i}}_{m_i} \delta K_i \Big\} J \mathrm{d}\xi$$

$$= -\int_I \Big\{ \delta \mathbf{r}' \cdot n_i \mathbf{d}_i + \delta \mathbf{d}_i \cdot (n_i \mathbf{r}' + \varepsilon_{kji} \tfrac{m_k}{2} \mathbf{d}_j') + \delta \mathbf{d}_j' \cdot \varepsilon_{ijk} \tfrac{m_i}{2} \mathbf{d}_k \Big\} \mathrm{d}\xi$$

variation of strain measures

$$\delta \Gamma_i = \frac{1}{J} (\delta \mathbf{r}' \cdot \mathbf{d}_i + \delta \mathbf{d}_i \cdot \mathbf{r}') \ , \quad \delta K_i = \frac{1}{2J} \varepsilon_{ijk} (\delta \mathbf{d}_k \cdot \mathbf{d}_j' + \delta \mathbf{d}_j' \cdot \mathbf{d}_k)$$

Internal virtual work

total strain energy stored in the beam

$$E = \int_I U(\xi,t)J(\xi)\,\mathrm{d}\xi = \int_I W\left(\Gamma_i(\xi,t), K_i(\xi,t);\xi\right)J(\xi)\,\mathrm{d}\xi$$

$$W(\Gamma_i, K_i;\xi) = \tfrac{1}{2}k_{\mathrm{t}}\left(K_1 - K_1^0\right)^2 + \tfrac{1}{2}k_{\mathrm{b}_2}\left(K_2 - K_2^0\right)^2 + \tfrac{1}{2}k_{\mathrm{b}_3}\left(K_3 - K_3^0\right)^2 + \\ + \tfrac{1}{2}k_{\mathrm{e}}\left(\lambda(\Gamma_i) - 1\right)^2 + \tfrac{1}{2}k_{\mathrm{s}_2}\Gamma_2^2 + \tfrac{1}{2}k_{\mathrm{s}_3}\Gamma_3^2$$

internal virtual work

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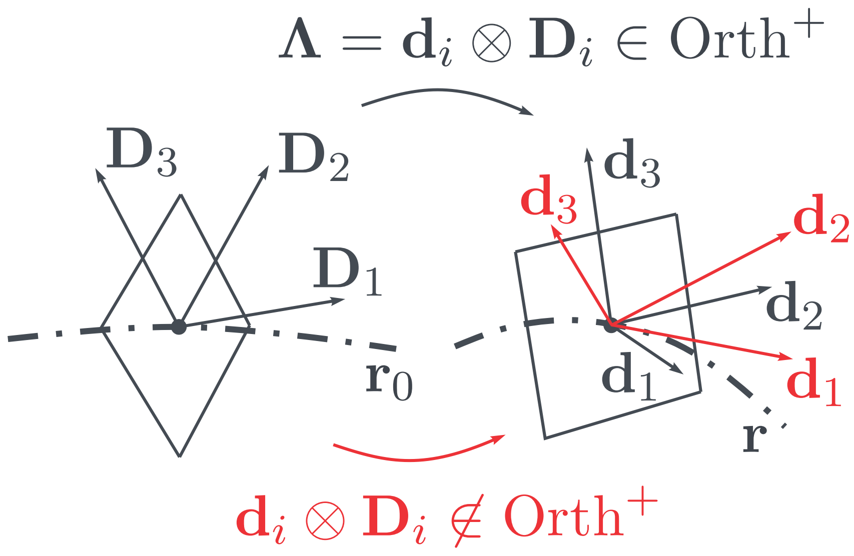
$$= -\int_I \left\{ \delta\mathbf{r}'\cdot n_i\mathbf{d}_i + \delta\mathbf{d}_i\cdot\left(n_i\mathbf{r}' + \varepsilon_{kji}\tfrac{m_k}{2}\mathbf{d}_j'\right) + \delta\mathbf{d}_j'\cdot\varepsilon_{ijk}\tfrac{m_i}{2}\mathbf{d}_k \right\}\mathrm{d}\xi$$

variation of strain measures

$$\delta\Gamma_i = \frac{1}{J}\left(\delta\mathbf{r}'\cdot\mathbf{d}_i + \delta\mathbf{d}_i\cdot\mathbf{r}'\right)\,,\quad \delta K_i = \frac{1}{2J}\varepsilon_{ijk}\left(\delta\mathbf{d}_k\cdot\mathbf{d}_j' + \delta\mathbf{d}_j'\cdot\mathbf{d}_k\right)$$

unconstrained variation of directors

Constraint virtual work



$$g_1 = \mathbf{d}_1 \cdot \mathbf{d}_1 - 1 = 0, \quad g_4 = \mathbf{d}_1 \cdot \mathbf{d}_2 = 0$$
$$g_2 = \mathbf{d}_2 \cdot \mathbf{d}_2 - 1 = 0, \quad g_5 = \mathbf{d}_1 \cdot \mathbf{d}_3 = 0$$
$$g_3 = \mathbf{d}_3 \cdot \mathbf{d}_3 - 1 = 0, \quad g_6 = \mathbf{d}_2 \cdot \mathbf{d}_3 = 0$$

orthonormality of directors

$$\delta W^{\mathrm{c}} = \sum_{k=1}^{6} \int_I g_k \delta\lambda_k \, \mathrm{d}\xi + \int_I \delta g_k \lambda_k \, \mathrm{d}\xi$$

$$g_7 = \Gamma_2 = \mathbf{d}_2 \cdot \mathbf{r}' = 0$$
$$g_8 = \Gamma_3 = \mathbf{d}_3 \cdot \mathbf{r}' = 0$$
$$g_9 = \Gamma_1 = \mathbf{d}_1 \cdot \mathbf{r}' - 1 = 0$$

shear-rigidity & inextensibility

$$g_7 = \Gamma_2 = \mathbf{d}_2 \cdot \mathbf{r}' = 0$$
$$g_8 = \Gamma_3 = \mathbf{d}_3 \cdot \mathbf{r}' = 0$$

shear-rigidity

$$\delta W^{\mathrm{c}} = \sum_{k=1}^{n_c} \int_I g_k \delta\lambda_k \, \mathrm{d}\xi + \int_I \delta g_k \lambda_k \, \mathrm{d}\xi$$

$n_c = 6$	shear-deformable
$n_c = 8$	shear-rigid
$n_c = 9$	inextensible & shear-rigid

External virtual work

$$\delta \mathbf{d}_i = \underline{\mathrm{ax}(\delta \mathbf{R}\mathbf{R}^{\mathrm{T}})} \times \mathbf{d}_i \quad \Rightarrow \delta \boldsymbol{\phi} = \tfrac{1}{2}\mathbf{d}_i \times \delta \mathbf{d}_i$$

virtual rotation

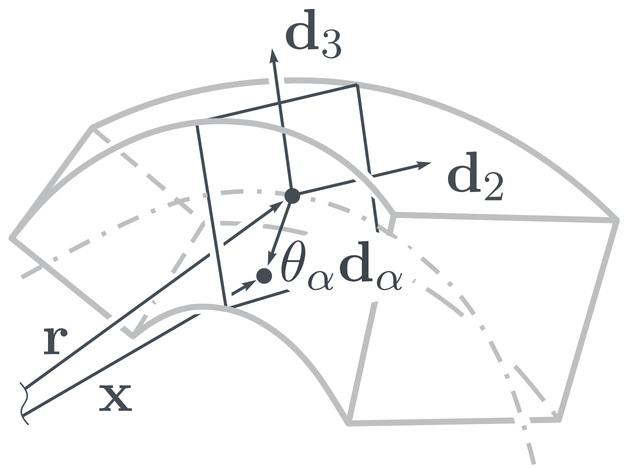
$= \delta \boldsymbol{\phi}$

$$\delta W^{\mathrm{ext}} = \int_I \{\delta \mathbf{r} \cdot \overline{\mathbf{n}} + \delta \boldsymbol{\phi} \cdot \overline{\mathbf{m}}\}\, J \,\mathrm{d}\xi + \sum_{k=0}^{1} \{\delta \mathbf{r} \cdot \overline{\mathbf{n}}_k + \delta \boldsymbol{\phi} \cdot \overline{\mathbf{m}}_k\}\big|_{\xi=\xi_k}$$

$\overline{\mathbf{n}}, \overline{\mathbf{m}}$ line distributed forces and moments

$\overline{\mathbf{n}}_k, \overline{\mathbf{m}}_k$ boundary forces and moments

Virtual work of inertia effects



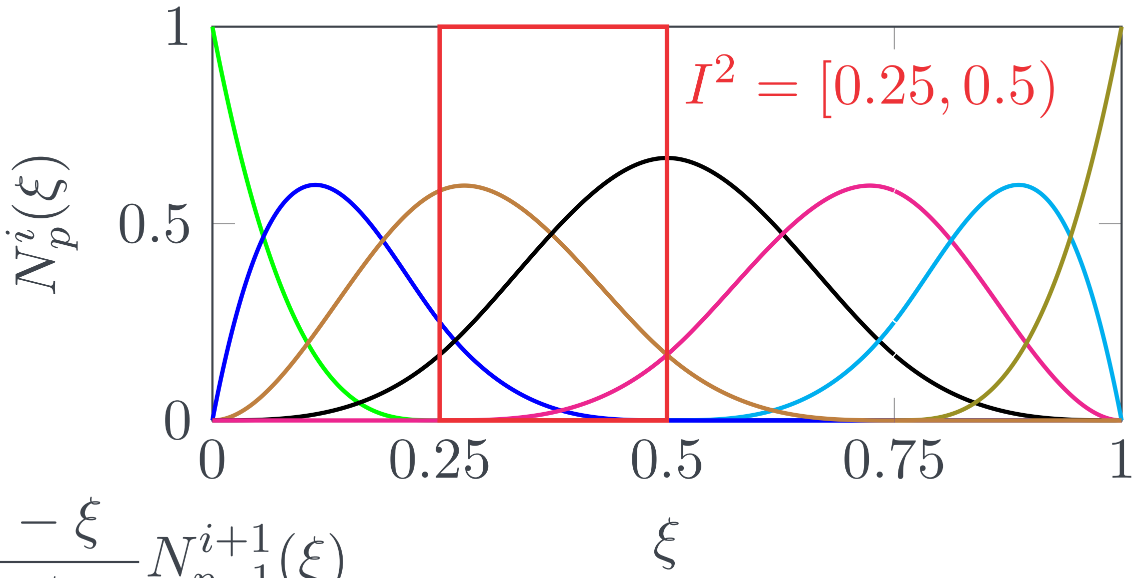
$$\delta W^{\mathrm{dyn}} = -\int_I \Big\{ \delta \mathbf{r} \cdot (A_{\rho_0} \ddot{\mathbf{r}} + B^{\alpha}_{\rho_0} \ddot{\mathbf{d}}_\alpha) + \delta \mathbf{d}_\alpha \cdot (B^{\alpha}_{\rho_0} \ddot{\mathbf{r}} + C^{\alpha\beta}_{\rho_0} \ddot{\mathbf{d}}_\beta) \Big\} J \,\mathrm{d}\xi$$

no $\ddot{\mathbf{d}}_1$ or $\delta \mathbf{d}_1$ contribution!

reference density

$$A_{\rho_0}(\xi) = \int_{A(\xi)} \rho_0(\xi)\,\mathrm{d}A\,,\quad B^{\alpha}_{\rho_0}(\xi) = \int_{A(\xi)} \rho_0(\xi)\theta_\alpha\,\mathrm{d}A\,,\quad C^{\alpha\beta}_{\rho_0}(\xi) = \int_{A(\xi)} \rho_0(\xi)\theta_\alpha\theta_\beta\,\mathrm{d}A$$

B-spline curves



B-spline shape functions

$$N_0^i(\xi) = \begin{cases} 1\,, & \xi \in [\xi_i, \xi_{i+1}) \\ 0\,, & \xi \notin [\xi_i, \xi_{i+1}) \end{cases}$$

$$N_p^i(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{p-1}^i(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{p-1}^{i+1}(\xi)$$

B-spline curves

connectivity matrix — generalized coordinates

$$_I\square(\xi, t) = \sum_{e=1}^{n} \chi_{I^e}(\xi) \mathbf{N}_p^e(\xi) \mathbf{C}_\square^e \mathbf{q}(t) \qquad \lambda_k^h(\xi, t) = \sum_{e=1}^{n} \chi_{I^e}(\xi) \mathbf{n}_p^e(\xi) \mathbf{C}_{\lambda_k}^e \boldsymbol{\lambda}_k(t)$$

$\mathbf{r}^h$ or $\mathbf{d}_i^h$ — shape function matrix

$$\mathbf{N}_p^e = \begin{pmatrix} N_p^e & \dots & N_p^{e+p} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & N_p^e & \dots & N_p^{e+p} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & N_p^e & \dots & N_p^{e+p} \end{pmatrix}$$

$$\mathbf{q}_\square^e = \mathbf{C}_\square^e \mathbf{q} = \begin{pmatrix} q_{\square 1}^e & \dots & q_{\square 1}^{e+p} & q_{\square 2}^e & \dots & q_{\square 2}^{e+p} & q_{\square 3}^e & \dots & q_{\square 3}^{e+p} \end{pmatrix}^{\mathrm{T}}$$

Principle of virtual work

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{dyn}} + \delta W^{\text{c}} \stackrel{!}{=} 0 \quad \forall \delta \mathbf{r}, \delta \mathbf{d}_i, \delta \boldsymbol{\lambda}, t$$

discrete virtual work contributions

$$\delta W^{\text{int,h}} = \delta \mathbf{q}^{\text{T}} \underline{\mathbf{f}^{\text{int}}(\mathbf{q})}$$ internal generalized forces

$$\delta W^{\text{ext,h}} = \delta \mathbf{q}^{\text{T}} \underline{\mathbf{f}^{\text{ext}}(t, \mathbf{q})}$$ external generalized forces

$$\delta W^{\text{dyn,h}} = -\delta \mathbf{q}^{\text{T}} \underline{\mathbf{M}} \ddot{\mathbf{q}}$$ (singular) mass matrix

$$\delta W^{\text{c,h}} = \delta \boldsymbol{\lambda}^{\text{T}} \underline{\boldsymbol{g}(\mathbf{q})} + \delta \mathbf{q}^{\text{T}} \underline{\mathbf{W}(\mathbf{q}) \boldsymbol{\lambda}}$$ integrated constraints, constraint forces

equations of motion

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} - \mathbf{f}^{\text{int}}(\mathbf{q}) - \mathbf{f}^{\text{ext}}(t, \mathbf{q}) - \mathbf{W}(\mathbf{q})\boldsymbol{\lambda} &= 0 \\ \mathbf{g}(\mathbf{q}) &= 0 \end{aligned}$$

differential algebraic equation of index 3, [1]

statics

$$\mathbf{R}(\mathbf{q}, \boldsymbol{\lambda}) = \begin{pmatrix} \mathbf{f}^{\text{int}}(\mathbf{q}) + \mathbf{f}^{\text{ext}}(\mathbf{q}) + \mathbf{W}(\mathbf{q})\boldsymbol{\lambda} \\ \boldsymbol{g}(\mathbf{q}) \end{pmatrix} = 0$$

[1] M. Arnold, O. Brüls. Convergence of the generalized-α scheme for constrained mechanical systems, 2007.

Numerical validation - cantilever beam

first and second elliptic integral

$$F(\theta,p) = \int_0^\theta (1-p^2\sin^2\tilde{\theta})^{-\frac{1}{2}}\,\mathrm{d}\tilde{\theta}\,, \quad E(\theta,p) = \int_0^\theta (1-p^2\sin^2\tilde{\theta})^{\frac{1}{2}}\,\mathrm{d}\tilde{\theta}$$

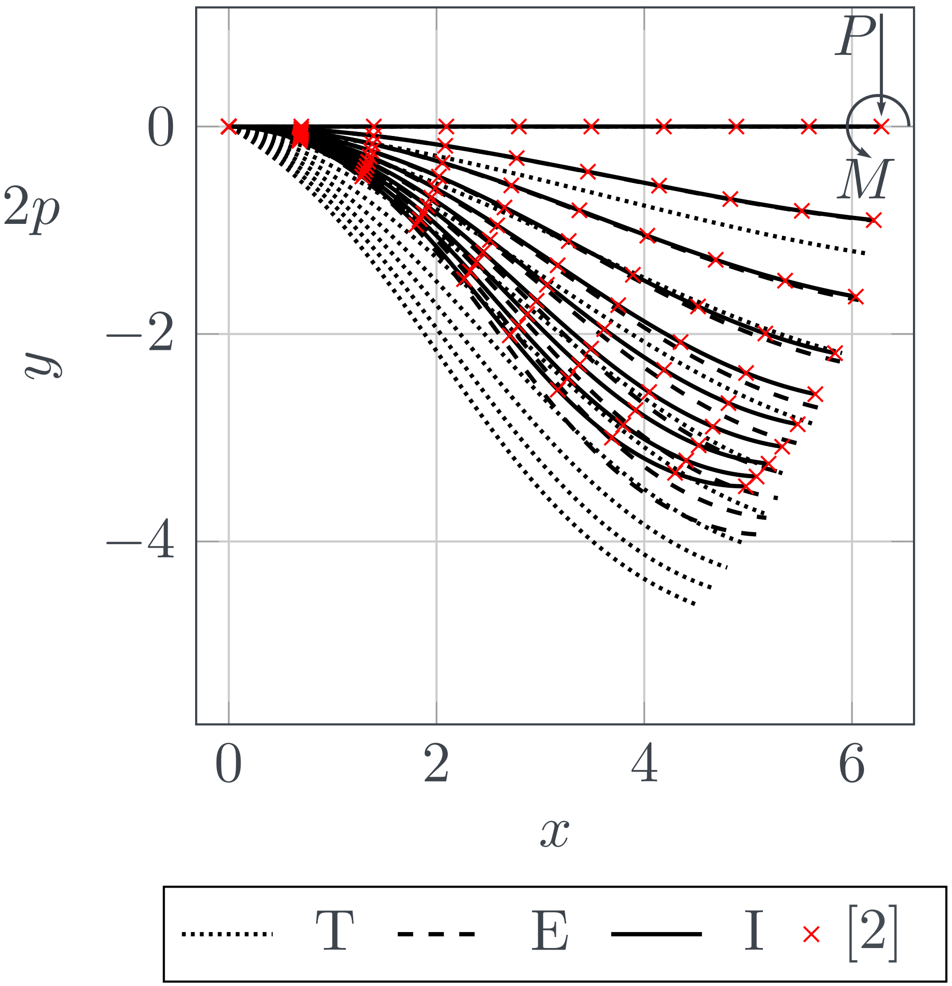
replace external moment M by force P
with rigid lever $e = M/P$

$$ks = F(\phi(s),p) - F(\phi_C,p)\,, \quad \cos\phi_C = ek/2p$$

$$k = \sqrt{P/k_{b_2}}$$

solve for p

inextensibility: $\sin\phi(L) = \sin\phi_B = 1/(p\sqrt{2})$



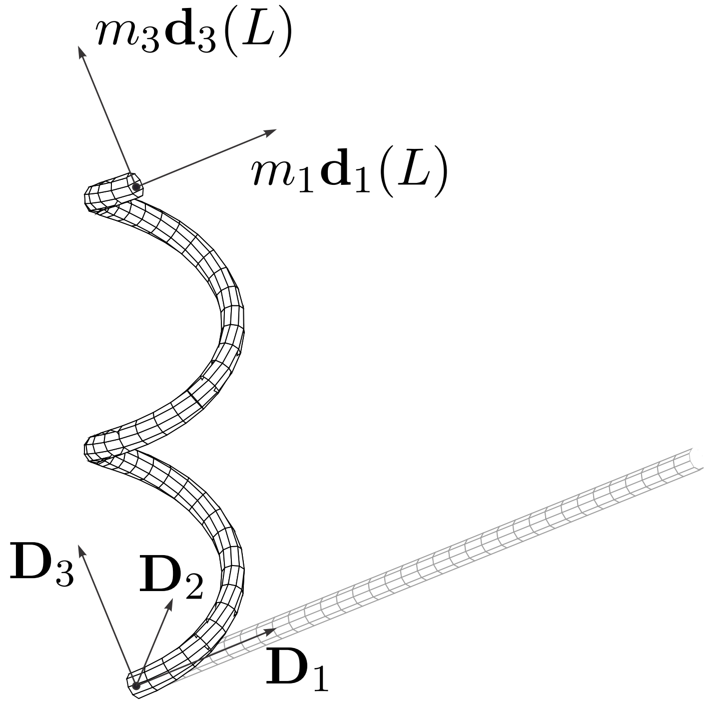
vertical and horizontal deflection

$$x(s) = 2p\big[\cos(\phi_B) - \cos(\phi)\big]/k$$

$$y(s) = -\big[2E(\phi_B,p) - 2E(\phi,p) + F(\phi,p) - F(\phi_B,p)\big]/k$$

[2] R. Frisch-Fay. Flexible Bars, 1962.

Numerical validation - helix



analytic solution (semi-inverse method)

$$\mathbf{r}^*(s) = R_0 \sin\alpha(s)\mathbf{e}_1 - R_0\cos\alpha(s)\mathbf{e}_2 + cR_0\alpha(s)\mathbf{e}_3$$

$$\alpha(s) = 2\pi n s/L$$

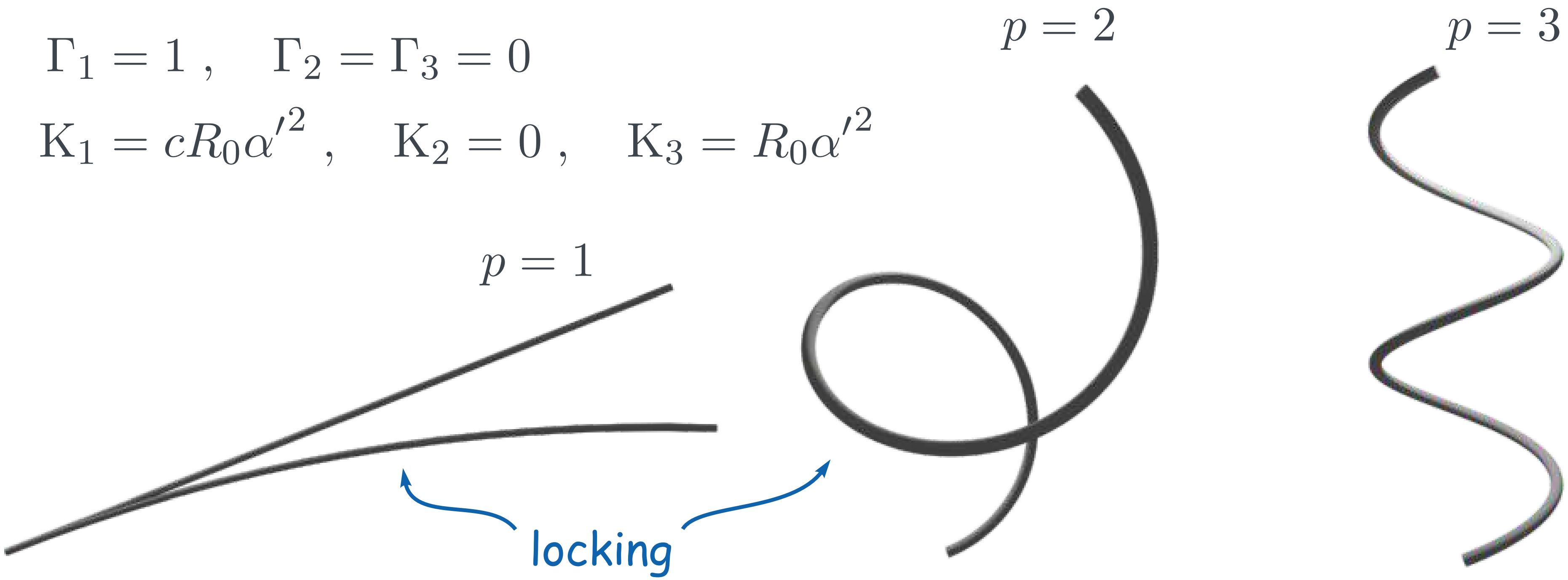
$$c = h/(R_0 2\pi n)$$

height radius #coils

strain measures

$$\Gamma_1 = 1\,, \quad \Gamma_2 = \Gamma_3 = 0$$

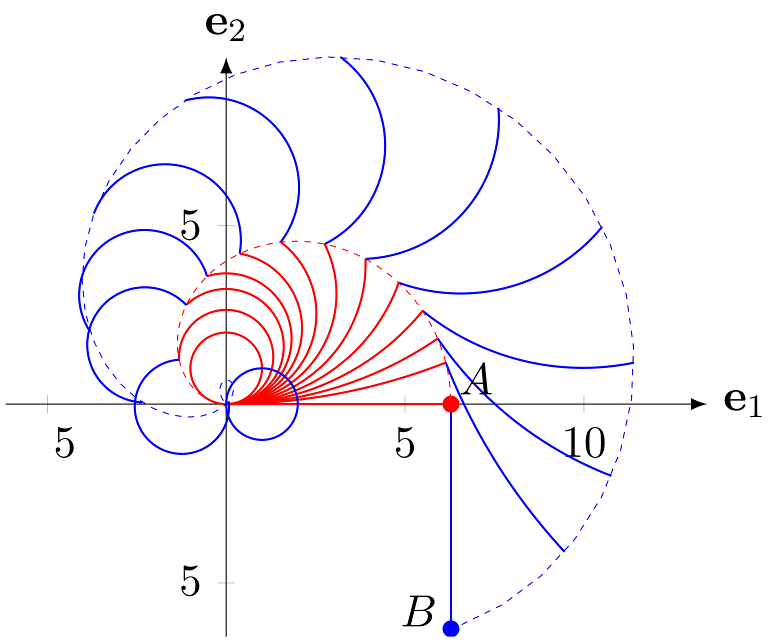
$$\mathrm{K}_1 = cR_0\alpha'^2\,, \quad \mathrm{K}_2 = 0\,, \quad \mathrm{K}_3 = R_0\alpha'^2$$

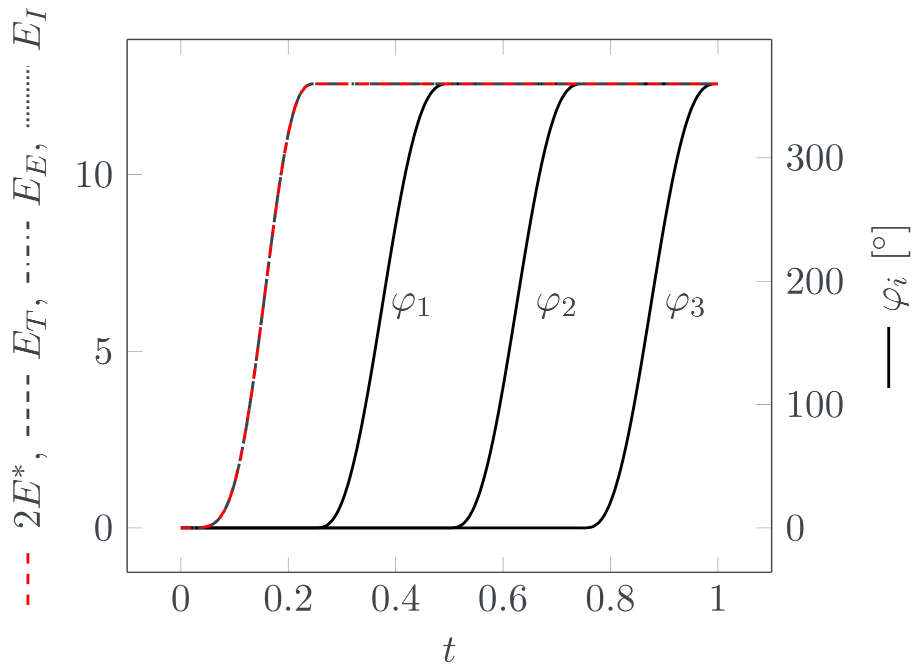


Numerical validation - objectivity

$$\Gamma_i^h = \frac{1}{J}\,\mathbf{r}^{h\prime}\cdot\mathbf{d}_i^h \qquad \xrightarrow[\mathbf{c}\in\mathbb{E}^3,\,\mathbf{Q}\in\mathrm{Orth}]{\mathbf{r}^{h,+} = \mathbf{Q}\mathbf{r}^h + \mathbf{c},\quad \mathbf{d}_i^{h,+} = \mathbf{Q}\mathbf{d}_i^h} \qquad \Gamma_i^{h,+} = \frac{1}{J}\,(\mathbf{r}^{h,+})'\cdot\mathbf{d}_i^{h,+} = \Gamma_i^h$$

$$K_i^h = \frac{1}{2J}\varepsilon_{ijk}\mathbf{d}_k^h\cdot\mathbf{d}_j^{h\prime} \qquad\qquad K_i^{h,+} = \frac{1}{2J}\varepsilon_{ijk}\mathbf{d}_k^{h,+}\cdot(\mathbf{d}_j^{h,+})' = K_i^h$$

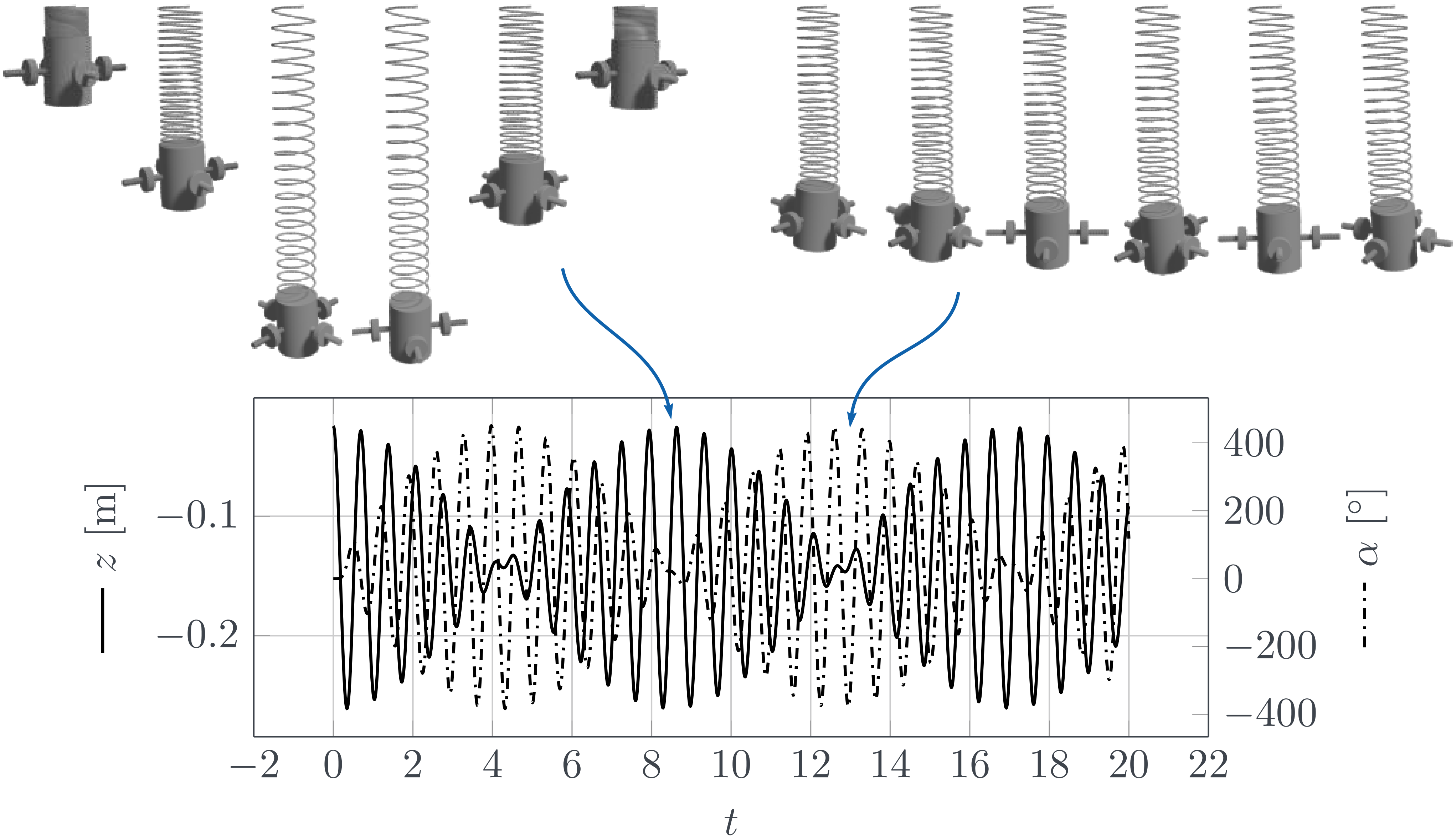




Numerical validation - Wilberforce pendulum

pure vertical oscillation

pure torsional oscillation



Conclusion

⊕

- rather straight forward finite element formulation
- shear-deformable, shear-rigid and inextensible shear-rigid beams
- large displacements and finite strains
- hyperelastic constitutive laws
- precurved reference configurations
- discretization preserves objectivity of strain measures

⊖

- DAE solver is required
- locking for very slender beams
- orthonormality of directors only weakly satisfied
- computationally expensive